ASYMPTOTIC BEHAVIOR OF THE CONTINUOUS AND DISCRETE SOLUTIONS TO A MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this paper we study the existence and asymptotic behavior of the solution $u(t)$ to the multi-term fractional differential equation

$$(*) \quad \partial_t^\alpha u(t) + \mu\partial_t^\beta u(t) = Au(t) + f(t), \quad t \geq 0,$$

where $1 \leq \beta \leq \alpha \leq 2$, $\mu \geq 0$, A is a closed and linear operator defined in a Banach space X, and for $\eta > 0$, $\partial_t^\eta u(t)$ is the Caputo fractional derivative of u.

To this end, we introduce a family of linear operators generated by A, we establish conditions for A to be the generator of such family, and investigate its asymptotic behavior to study the behavior of u as t tends to infinity.

Furthermore, we analyze a discrete version of $(*)$ and introduce a sequence of linear operators generated by A to explore its connection with the continuous solution $u(t)$ and the discrete solution u^n of this equation. Finally, we derive an error estimate for $\|u(t_n) - u^n\|$ and provide examples to illustrate our results.

1. Introduction

Consider a rigid plate of mass m and area S. Assume that the plate is immersed in a newtonian fluid of infinite extend and suppose that it is connected to a fixed point by a massless spring of stiffness σ. If ρ and ν denote, respectively, the fluid density and viscosity, then the displacement $u(t)$ of the plate at time t, obeys the Bagley-Torvik equation

$$mu''(t) + 2S\sqrt{\nu}\rho\partial_t^{\frac{3}{2}}u(t) + \sigma u(t) = 0, \tag{1.1}$$

subject to initial conditions $u(0) = u_0$ and $u'(0) = u_1$. See for instance [35]. Here, $\partial_t^{\frac{3}{2}}u$ denotes the Caputo fractional derivative of order $\frac{3}{2}$ of u (see [28]). The existence of exact and numerical solutions to the scalar multi-term equation (1.1) has been extensively studied in recent years. For instance, [3, 4, 6, 9, 36, 37, 38] investigate numerical methods for the Bagley-Torvik equation using various approaches, while [2, 7, 19, 26, 31, 32, 39] focus on the stability and numerical solutions of multi-term fractional differential equations with arbitrary fractional orders.

In a more general context, this equation can be written in an abstract form as

$$\begin{cases} \partial_t^\alpha u(t) + \mu\partial_t^\beta u(t) &= Au(t) + f(t), \quad t \geq 0, \\ u(0) &= u_0, \\ u'(0) &= u_1, \end{cases} \tag{1.2}$$

where $1 \leq \beta \leq \alpha \leq 2$, $\mu \geq 0$, f is a given function, A is a closed linear operator defined in a Banach space X, $u_0, u_1 \in X$, and, $\partial_t^\alpha, \partial_t^\beta$ are the Caputo fractional derivatives of order α and β, respectively.

The existence of mild solutions to abstract multi-term fractional differential equations in the form of (1.2) represents a subject of increasing interest in the last years and the typical method to find these solutions consists in the construction of a strongly continuous family of operators whose properties are analogous to the C_0-semigroups of operators. See for instance [1, 17, 21, 34, 39] and references therein.

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In this work, we investigate the existence of solutions to the abstract multi-term equation (1.2). Our approach is based on the theory of fractional resolvent families, which enables the representation of the solution to (1.2) as a variation-of-constants formula involving these resolvent families. More concretely, we show that the mild solution to (1.2) is given by

$$u(t)=S_{\alpha,\beta,\mu}(t)u_0+\mu(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})(t)u_0+(g_1*S_{\alpha,\beta,\mu})(t)u_1+\mu(g_{\alpha-\beta+1}*S_{\alpha,\beta,\mu})(t)u_1+(g_{\alpha-1}*S_{\alpha,\beta,\mu}*f)(t),$$

where $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$ is a strongly continuous family whose Laplace transform verifies $\hat{S}_{\alpha,\beta,\mu}(\lambda)x = \lambda^{\alpha-1}(\lambda^\alpha+\mu\lambda^\beta-A)^{-1}x$ for all $x\in X$ and $\lambda\in\mathbb{C}$ with $\lambda^\alpha+\mu\lambda^\beta\in\rho(A)$ and for $\eta>0$ the function g_η is defined by $g_\eta(t):=\frac{t^{\eta-1}}{\Gamma(\eta)}$, $t>0$. See Definition 2.1 below.

Moreover, we give a discretization in time to equation (1.2) based on a sequence of linear operators generated by A and the backward Euler convolution method for $\tau>0$ (see for instance [12, 13, 14, 24, 25]), to study the existence of solutions to the abstract discrete multi-term equation

$$ {}_C\nabla^\alpha u^n+\mu\, {}_C\nabla^\beta u^n = Au^n+f^n, \tag{1.3}$$

for all $n\in\mathbb{N}_0$, under the initial conditions $u^0=u_0, u^1=u_1$. Here, ${}_C\nabla^\alpha u^n$ represents an approximation of the Caputo fractional derivative $\partial_t^\alpha u(t)$ at time $t=\tau n$ (where $\tau>0$ is the step size) which is defined by

$$ {}_C\nabla^\alpha u^n := \sum_{j=2}^{n} k_\tau^{2-\alpha}(n-j)\frac{(u^j-2u^{j-1}+u^{j-2})}{\tau^2},$$

where, for $\rho_j^\tau(t):=e^{-\frac{t}{\tau}}\left(\frac{t}{\tau}\right)^j\frac{1}{\tau j!}$, u^j is defined by $u^j:=\int_0^\infty \rho_j^\tau(t)u(t)dt$, and $k_\tau^\eta(j):=\frac{\tau^{\eta-1}\Gamma(\eta+j)}{\Gamma(\eta)\Gamma(j+1)}$ for all $j\in\mathbb{N}_0$ and $\eta>0$. It is a well-known fact that u^n approximates the value $u(t_n)$ where $t_n:=n\tau$, and the solution to (1.3) can be written again as a variation-of-parameters formula as (see Theorem 3.17 below)

$$u^n=S^n_{\alpha,\beta,\mu}u_0+\mu\tau(k_\tau^{\alpha-\beta}\star S_{\alpha,\beta,\mu})^n u_0+\tau(k_\tau^1\star S_{\alpha,\beta,\mu})^n u_1+\mu\tau(k_\tau^{\alpha-\beta+1}\star S_{\alpha,\beta,\mu})^n u_1+\tau^2(k_\tau^{\alpha-1}\star S_{\alpha,\beta,\mu}\star f)^n,$$

for all $n\geq 2$, where $S^n_{\alpha,\beta,\mu}$ is defined as

$$S^n_{\alpha,\beta,\mu}x:=\int_0^\infty \rho_n^\tau(t)S_{\alpha,\beta,\mu}(t)xdt,$$

for all $x\in X$, and for $\eta>0$,

$$(k_\tau^\eta\star S_{\alpha,\beta,\mu})^n x:=\sum_{j=0}^{n}k_\tau^\eta(n-j)S^j_{\alpha,\beta,\mu}x.$$

Finally, we study the we study the difference $\|u(t_n)-u^n\|$, where u is the solution to (1.2) and u^n solves the discrete equation (1.3) and we show that, given a suitable conditions on the parameters α,β and μ, there exists a constant $C=C(T)>0$ (independent of the solution, the data and the step size) such that, for $0<t_n\leq T$, there holds

$$\|u(t_n)-u^n\|\leq C\tau t_n^{\beta\varepsilon-1}\left(\|A^\varepsilon u_0\|+\|A^\varepsilon u_1\|+\|A^\varepsilon f\|\right),$$

where $0<\varepsilon<1$ satisfies $\beta\varepsilon<1$ and u_0,u_1 and $f(t)$ belong to the domain of A^ε.

The paper is organized as follows. In Section 2 we give preliminaries on resolvent families and sequences. Section 3 is devoted to the existence of solutions to the discrete multi-term equation (1.3). Here, given a time step size $\tau>0$, we study the connection between the continuous and the discrete resolvent families $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$ and $\{S^n_{\alpha,\beta,\mu}\}_{n\in\mathbb{N}}$, respectively, as well as, its consequences on the existence of solutions to (1.3). In Section 4 we study error estimates of the continuous and discrete solution, that is, we study the norm difference $\|u(t_n)-u^n\|$. Additionally, in Section 5 we give some examples to illustrate the theoretical results. Finally, Section 6 corresponds to an Appendix that summarizes the main properties of resolvent families.

2. Resolvent families, mild solutions and fractional calculus.

For a Banach space $X \equiv (X, \|\cdot\|)$, $\mathcal{B}(X)$ denotes the Banach space of all bounded and linear operators from X into X. Given a closed linear operator A defined on X, its resolvent set is denoted by $\rho(A)$, the resolvent operator is defined by $R(\lambda, A) = (\lambda - A)^{-1}$ for all $\lambda \in \rho(A)$, and $\sigma(A)$ defines the spectrum of A. A family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called *exponentially bounded* if there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$, for any $t \geq 0$. We observe that if $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is exponentially bounded, then the Laplace transform of $S(t)$, $\hat{S}(\lambda)x := \int_0^\infty e^{-\lambda t}S(t)xdt$, exists for all $\mathrm{Re}\lambda > \omega$.

Definition 2.1. *Let $\mu \geq 0$, and $1 \leq \beta \leq \alpha \leq 2$ be given. Let A be a closed linear operator defined in a Banach space X. The operator A is called the generator of an (α, β, μ)*-resolvent family *if there exist $\omega \geq 0$ and a strongly continuous and exponentially bounded function $S_{\alpha,\beta,\mu} : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha + \mu\lambda^\beta : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-1}(\lambda^\alpha + \mu\lambda^\beta - A)^{-1}x = \int_0^\infty e^{-\lambda t}S_{\alpha,\beta,\mu}(t)xdt, \quad \textit{for all} \quad \mathrm{Re}\lambda > \omega, x \in X.$$

In this case, $\{S_{\alpha,\beta,\mu}(t)\}$ is called the (α, β, μ)-resolvent family generated by A.

Let a and b defined, respectively, by

$$a(t) := \int_0^t g_{\alpha-1}(t-s)b(s)ds, \quad b(t) := E_{\alpha-\beta,1}(-\mu t^{\alpha-\beta}),$$

where, for $\nu > 0$, $g_\nu(t) := \frac{t^{\nu-1}}{\Gamma(\nu)}$, and $E_{\nu,1}$ denotes the Mittag-Leffler function. Then, $\hat{a}(\lambda) = \frac{1}{\lambda^\alpha+\mu\lambda^\beta}$, $\hat{b}(\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^\alpha+\mu\lambda^\beta}$ for all $\mathrm{Re}(\lambda) > 0$, and $\{S_{\alpha,\beta,\mu}(t)\}$ corresponds to an (a, b)-regularized families generated by A, see [20], and from [20, Lemma 2.2 and Proposition 2.5], it has the following properties.

Proposition 2.2. *Let $\mu \geq 0$, and $1 \leq \beta \leq \alpha \leq 2$ be given. Let $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$ be the (α, β, μ)-resolvent family generated by A. Then,*

(1) *$S_{\alpha,\beta,\mu}(0) = I$, where I denotes the identity operator in X.*
(2) *For all $x \in D(A)$ and $t \geq 0$ we have $S_{\alpha,\beta,\mu}(t)x \in D(A)$ and $AS_{\alpha,\beta,\mu}(t)x = S_{\alpha,\beta,\mu}(t)Ax$.*
(3) *For $x \in X$ and $t \geq 0$ we have $\int_0^t a(t-s)S_{\alpha,\beta,\mu}(s)xds \in D(A)$ and*

$$S_{\alpha,\beta,\mu}(t)x = b(t)x + A\int_0^t a(t-s)S_{\alpha,\beta,\mu}(s)xds.$$

For further details on resolvent families, see Appendix in Section 6.

Definition 2.3. *We say that a function $u \in C^1(\mathbb{R}_+, X)$ is a* strong solution *to equation* (1.2) *if $u(t) \in D(A)$ for all $t \geq 0$ and satisfies* (1.2).

If we take Laplace transform in (1.2) we obtain

$$(\lambda^\alpha + \mu\lambda^\beta - A)\hat{u}(\lambda) = (\lambda^{\alpha-1} + \mu\lambda^{\beta-1})u_0 + (\lambda^{\alpha-2} + \mu\lambda^{\beta-2})u_1 + \hat{f}(\lambda),$$

for all $\mathrm{Re}(\lambda) > 0$. If $\lambda^\alpha + \mu\lambda^\beta \in \rho(A)$, then

$$\begin{aligned}\hat{u}(\lambda) \;=\; & \lambda^{\alpha-1}(\lambda^\alpha + \mu\lambda^\beta - A)^{-1}u_0 + \mu\lambda^{\beta-1}(\lambda^\alpha + \mu\lambda^\beta - A)^{-1}u_0 \\ & +\lambda^{\alpha-2}(\lambda^\alpha + \mu\lambda^\beta - A)^{-1}u_1 + \mu\lambda^{\beta-2}(\lambda^\alpha + \mu\lambda^\beta - A)^{-1}u_1 + (\lambda^\alpha + \mu\lambda^\beta - A)^{-1}\hat{f}(\lambda),\end{aligned}$$

where $u_0, u_1 \in X$. The uniqueness of the Laplace transform and Definition 2.1 imply that if A is the generator of a resolvent family $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$, then a *solution* to Problem (1.2) is given by

(2.1)
$$u(t) = S_{\alpha,\beta,\mu}(t)u_0+\mu(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})(t)u_0+(g_1*S_{\alpha,\beta,\mu})(t)u_1+\mu(g_{\alpha-\beta+1}*S_{\alpha,\beta,\mu})(t)u_1+(g_{\alpha-1}*S_{\alpha,\beta,\mu}*f)(t).$$

As u_0, u_1 merely belong to X, we can not prove (by Proposition 2.2) that $u(t)$ defined by (2.1) belongs to $D(A)$ for all $t \geq 0$ to obtain a strong solution, and therefore we need to introduce the following notion of solution.

Definition 2.4. *We say that a continuous function* $u : \mathbb{R}_+ \to X$ *is a* mild solution *to equation* (1.2) *if* $u(t)$ *satisfies* (2.1) *for all* $t \geq 0$.

Now, consider $f(t) = 0$ for all $t \geq 0$, and assume that $\alpha - \beta > 1$. As $S_{\alpha,\beta,\mu}(0)x = x$, $\hat{S}_{\alpha,\beta,\mu}(\lambda)x = \lambda^{\alpha-1}(\lambda^\alpha + \mu\lambda^\beta - A)^{-1}x$ and $(\lambda^\alpha + \mu\lambda^\beta - A)(\lambda^\alpha + \mu\lambda^\beta - A)^{-1}x = x$, for all $x \in X$, we obtain for any $\lambda \in \mathbb{C}$ with $\lambda^\alpha + \mu\lambda^\beta \in \rho(A)$, that

$$\widehat{S'}_{\alpha,\beta,\mu}(\lambda)x = \lambda\hat{S}_{\alpha,\beta,\mu}(\lambda)x - x = \lambda^\alpha(\lambda^\alpha + \mu\lambda^\beta - A)^{-1}x - x = \frac{1}{\lambda^{\alpha-1}}A\hat{S}_{\alpha,\beta,\mu}(\lambda)x - \mu\frac{1}{\lambda^{\alpha-\beta-1}}\hat{S}_{\alpha,\beta,\mu}(\lambda)x,$$

and therefore

$$S'_{\alpha,\beta,\mu}(t)x = A(g_{\alpha-1} * S_{\alpha,\beta,\mu})(t)x - \mu(g_{\alpha-\beta-1} * S_{\alpha,\beta,\mu})(t)x, \quad t \geq 0, x \in X. \tag{2.2}$$

Thus, if a mild solution u to equation (1.2) is differentiable on $\mathbb{R}_+$, then by (2.2), it verifies $u(0) = u_1$ and $u'(0) = u_1$, and therefore, in this case, a mild solution is a strong solution of (1.2).

Now, we recall the definition of Caputo fractional derivative. For $\alpha > 0$, let $m = \lceil\alpha\rceil$ be the smallest integer m greater than or equal to α. Let $f : \mathbb{R}_+ \to X$ be a C^m-differentiable function. The *Caputo fractional derivative of order* α is defined by $\partial_t^\alpha f(t) := \int_0^t g_{m-\alpha}(t-s)f^{(m)}(s)ds$. It is well known that if $\alpha = m \in \mathbb{N}$, then $\partial_t^m f = \frac{d^m f}{dt^m}$, and that if $1 < \alpha < 2$, then $\widehat{\partial_t^\alpha f}(\lambda) = \lambda^\alpha\hat{f}(\lambda) - \lambda^{\alpha-1}f(0) - \lambda^{\alpha-2}f'(0)$. For more details on fractional calculus, we refer to [18].

The operator $A : D(A) \subset X \to X$ is called *sectorial of angle* θ if there are constants $\omega \in \mathbb{R}$, $M > 0$ and $\theta \in (\pi/2, \pi)$ such that $\rho(A) \supset S_{\theta,\omega} := \{z \in \mathbb{C} : z \neq \omega : |\arg(z-\omega)| < \theta\}$ and

$$\|(z-A)^{-1}\| \leq \frac{M}{|z-\omega|} \quad \text{for all} \quad z \in S_{\theta,\omega}.$$

In this case, we write $A \in \mathrm{Sect}(\theta, \omega, M)$. We may assume, without lost of generality, that $\omega = 0$. In fact, if not so we can take the operator $A - \omega I$, which is also sectorial. In that case, we write $A \in \mathrm{Sect}(\theta, M)$ and we denote the sector $S_{\theta,0}$ as S_θ. More details on sectorial operators can be found in [8, 15].

Let A be a closed operator whose resolvent set contains the real axis $(-\infty, 0]$. For $0 \leq \varepsilon \leq 1$, X^ε denotes the domain of the fractional power A^ε, that is $X^\varepsilon := D(A^\varepsilon)$ endowed with the graph norm $\|x\|_\varepsilon = \|A^\varepsilon x\|$. Examples of such operators are sectorial operators with $\omega \geq 0$. It is a well known fact that if $0 < \varepsilon < 1$, and $x \in D(A)$, then there exists a constant $\kappa \equiv \kappa_\varepsilon > 0$ such that (see [27])

$$\|A^\varepsilon x\| \leq \kappa\|Ax\|^\varepsilon\|x\|^{1-\varepsilon}. \tag{2.3}$$

The set of non-negative integer numbers is denoted by $\mathbb{N}_0$ and the non-negative real numbers by $\mathbb{R}_0^+$. Take $\tau > 0$ fixed and $n \in \mathbb{N}_0$. We define the function ρ_n^τ by $\rho_n^\tau(t) := e^{-\frac{t}{\tau}}\left(\frac{t}{\tau}\right)^n\frac{1}{\tau n!}$. We notice that $\rho_n^\tau(t) \geq 0$ for all $t \geq 0$, $n \in \mathbb{N}_0$, and $\int_0^\infty \rho_n^\tau(t)dt = 1$, for all $n \in \mathbb{N}_0$.

Given a bounded and locally integrable function $u : \mathbb{R}_0^+ \to X$, we define the sequence $(u^n)_n$ by

$$u^n := \int_0^\infty \rho_n^\tau(t)u(t)dt, \quad n \in \mathbb{N}_0. \tag{2.4}$$

The vector space of all vector-valued functions $v : \mathbb{R}_0^+ \to X$ is denoted by $\mathcal{F}(\mathbb{R}_0^+; X)$. The *backward Euler operator* $\nabla_\tau : \mathcal{F}(\mathbb{R}_0^+; X) \to \mathcal{F}(\mathbb{R}_0^+; X)$ is defined by $\nabla_\tau v^n := \frac{v^n - v^{n-1}}{\tau}$, $n \in \mathbb{N}$. For $m \geq 2$, $\nabla_\tau^m : \mathcal{F}(\mathbb{R}_0^+; X) \to \mathcal{F}(\mathbb{R}_0^+; X)$ is defined recursively as

$$(\nabla_\tau^m v)^n := \nabla_\tau^{m-1}(\nabla_\tau v)^n, \quad n \geq m,$$

where $\nabla_\tau^1 \equiv \nabla_\tau$ and ∇_τ^0 is the identity operator.

In order to define the fractional difference operators, we introduce the sequence (see [22])

$$k_\tau^\alpha(n) := \int_0^\infty \rho_n^\tau(t)g_\alpha(t)dt, \quad n \in \mathbb{N}_0, \alpha > 0.$$

From definition, it follows that $k_\tau^\alpha(n) = \frac{\tau^{\alpha-1}\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}$ for any $n \in \mathbb{N}_0$, and $\alpha > 0$.

Definition 2.5. [5, 29] *Let $\alpha > 0$. Give a vector-valued sequence $v \in \mathcal{F}(\mathbb{R}; X)$, the $\alpha^{\text{th}}-$fractional sum of v defined by $(\nabla_\tau^{-\alpha} v)^n := \tau \sum_{j=0}^n k_\tau^\alpha(n-j)v^j$, $n \in \mathbb{N}_0$.*

Definition 2.6. [5, 29] *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. The* Caputo fractional backward difference operator of order α, ${}_C\nabla^\alpha : \mathcal{F}(\mathbb{R}_+; X) \to \mathcal{F}(\mathbb{R}_+; X)$, *is defined by $({}_C\nabla^\alpha v)^n := \nabla_\tau^{-(m-\alpha)}(\nabla_\tau^m v)^n$, $n \in \mathbb{N}$, where $m-1 < \alpha < m$.*

In this definition, if $\alpha \in \mathbb{N}_0$, then ${}_C\nabla^\alpha$ is defined as the backward difference operator ∇_τ^α, and we adopt the convention $\sum_{j=0}^{-k} v^j = 0$, for all $k \in \mathbb{N}$ (see [14, Chapter 1, Section 1.5]).

The following result can be obtained similarly to [29, Theorem 2.7], and relates the Caputo fractional derivative and the Caputo fractional backward difference operator.

Theorem 2.7. *Let $1 < \alpha < 2$. If $u : [0, \infty) \to X$ is a twice differentiable and bounded function, then $\int_0^\infty \rho_n^\tau(t)\partial_t^\alpha u(t)dt = {}_C\nabla^\alpha u^n$, for all $n \geq 2$, where $(u^n)_n$ defines the sequence* (2.4)*.*

Additionally, the next Lemma gives an expression for the Z-transform to the Caputo fractional backward difference operator, which is an analogous result for the Laplace transform of the Caputo fractional derivative. It proofs follows similarly to [5, Theorem 3.12].

Lemma 2.8. *Let $1 < \alpha < 2$. Let $u : [0, \infty) \to X$ be a twice differentiable and bounded function. Define $(u^n)_n$ by the sequence* (2.4)*. If $w^n := {}_C\nabla^\alpha u^n$, $n \in \mathbb{N}$, then*

$$\tilde{w}(z) = \frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^\alpha \tilde{u}(z) - \frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^{\alpha-1} u(0) - \frac{1}{\tau^{\alpha-1}}\left(\frac{z-1}{z}\right)^{\alpha-2} u'(0).$$

For a given family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$, we define the sequence $S^n x := \int_0^\infty \rho_n^\tau(t)S(t)xdt$, for any $n \in \mathbb{N}_0, x \in X$. For a continuous and bounded function $c : \mathbb{R}_+ \to \mathbb{C}$ we also define $c^n := \int_0^\infty \rho_n^\tau(t)c(t)dt$, $n \in \mathbb{N}_0$, and the discrete convolution $c \star S$ as $(c \star S)^n := \sum_{k=0}^n c^{n-k}S^k, n \in \mathbb{N}_0$.

The following results will be useful to prove the existence of solutions to (1.3).

Theorem 2.9. [29] *Let $c : \mathbb{R}_+ \to \mathbb{C}$ be Laplace transformable such that $\hat{c}(1/\tau)$ exists, and let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1/\tau)$ exists. Then, for all $x \in X$, $\int_0^\infty \rho_n^\tau(t)(c * S)(t)xdt = \tau(c \star S)^n x, n \in \mathbb{N}_0$.*

Proposition 2.10. *Let $\alpha > 0$. Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1/\tau)$ exists. Then, $\int_0^\infty \rho_n^\tau(t)(g_\alpha * S)(t)xdt = \tau\sum_{j=0}^n k_\tau^\alpha(n-j)S^j x$, for all $x \in X$ and $n \in \mathbb{N}_0$.*

In particular, we have that for any $\alpha, \beta > 0$,

$$k_\tau^{\alpha+\beta}(n) = \tau\sum_{j=0}^n k_\tau^\alpha(n-j)k_\tau^\beta(j) = \tau(k_\tau^\alpha \star k_\tau^\beta)^n, \quad n \in \mathbb{N}_0. \tag{2.5}$$

By $s(\mathbb{N}_0, X)$, we denote the vector space consisting of all sequences $s : \mathbb{N}_0 \to X$. Given a vector-valued sequence $s \in s(\mathbb{N}_0, X)$, its Z-transform, $\tilde{s}$, is defined by $\tilde{s}(z) := \sum_{j=0}^\infty z^{-j}s(j)$, where $z \in \mathbb{C}$. The convergence of this series holds for $|z| > R$, where R is large enough, and if $s_1, s_2 \in s(\mathbb{N}_0, X)$ and $\tilde{s_1}(z) = \tilde{s_2}(z)$ for all $|z| > R$ for some $R > 0$, then $s_1(j) = s_2(j)$ for all $j = 0, 1, ...$

Definition 2.11. *Let $\mu \geq 0$, and $1 \leq \beta \leq \alpha \leq 2$ be given. The closed linear operator A is called the generator of the (α, β, μ)-resolvent sequence $\{S_{\alpha,\beta,\mu}^n\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ if it satisfies the following conditions*

(1) *$S_{\alpha,\beta,\mu}^n x \in D(A)$ for all $x \in X$ and $AS_{\alpha,\beta,\mu}^n x = S_{\alpha,\beta,\mu}^n Ax$ for all $x \in D(A)$, and $n \in \mathbb{N}_0$.*
(2) *For each $x \in X$ and $n \in \mathbb{N}_0$,*

$$S_{\alpha,\beta,\mu}^n x = b^n x + \tau A(a \star S_{\alpha,\beta,\mu})^n x = b^n x + \tau A\sum_{j=0}^n a^{n-j}S_{\alpha,\beta,\mu}^j x, \tag{2.6}$$

where $a^m := \tau\sum_{j=0}^m k_\tau^{\alpha-1}(m-j)b^j$ and $b^j := \sum_{l=0}^\infty k_\tau^{(\alpha-\beta)l+1}(j)(-\mu)^l$.

Remark 2.12. Let $b(t) := E_{\alpha-\beta,1}(-\mu t^{\alpha-\beta})$ and $a(t) := (g_{\alpha-1} * b)(t)$. By [16, Formula 11.15] *and Proposition 2.10, we have*

$$b^j = \int_0^\infty \rho_n^\tau(t)b(t)dt = \sum_{l=0}^{\infty} k_\tau^{(\alpha-\beta)l+1}(j)(-\mu)^l \quad \text{and} \quad a^j = \int_0^\infty \rho_n^\tau(t)a(t)dt = \tau\sum_{j=0}^{m} k_\tau^{\alpha-1}(m-j)b^j.$$

Proposition 2.13. *Let $\{S^n_{\alpha,\beta,\mu}\}_{n\in\mathbb{N}_0} \subset B(X)$ be a discrete (α,β,μ)-resolvent sequence generated by A. Then its Z-transform satisfies*

$$\widetilde{S}_{\alpha,\beta,\mu}(z)x = \frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^{\alpha-1}\left(\frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^{\alpha} + \mu\frac{1}{\tau^\beta}\left(\frac{z-1}{z}\right)^{\beta} - A\right)^{-1} x, \quad x\in X,\ |z|>1.$$

Proof. Let $x \in X$ and $|z| > 1$. Taking Z-transform in (2.6), we obtain

$$\left(\frac{1}{\tau\tilde{a}(z)} - A\right)\widetilde{S}_{\alpha,\beta,\mu}(z)x = \frac{\tilde{b}(z)}{\tau\tilde{a}(z)}x.$$

As A is a closed operator, by (1) in Definition 2.11, we deduce that $1/\tau\tilde{a}(z) \in \rho(A)$ and

$$\widetilde{S}_{\alpha,\beta,\mu}(z)x = \frac{\tilde{b}(z)}{\tau\tilde{a}(z)}\left(\frac{1}{\tau\tilde{a}(z)} - A\right)^{-1} x. \tag{2.7}$$

Let $b(t)$ and $a(t)$ as in Remark 2.12. By [11, Proposition 2.1], $\tilde{b}(z) = \frac{1}{\tau}\hat{b}\left(\frac{z-1}{\tau z}\right)$. Additionally, as $\hat{b}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha + \mu\lambda^\beta)^{-1}$ (see for instance [16]), we obtain

$$\tilde{b}(z) = \frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^{\alpha-1}\left(\frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^{\alpha} + \mu\frac{1}{\tau^\beta}\left(\frac{z-1}{z}\right)^{\beta}\right)^{-1}.$$

Finally, by Proposition 2.10, $a^n = \tau(k_\tau^{\alpha-1} \star b)^n$, and thus

$$\tilde{a}(z) = \tau\tilde{k}_\tau^{\alpha-1}(z)\tilde{b}(z) = \frac{1}{\tau}\left(\frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^{\alpha} + \mu\frac{1}{\tau^\beta}\left(\frac{z-1}{z}\right)^{\beta}\right)^{-1},$$

and the result follows from (2.7). □

From Remark 2.12 we have the following result.

Proposition 2.14. *Let $\mu \geq 0$, and $1 \leq \beta \leq \alpha \leq 2$ be given. Assume that A is the generator of an (α,β,μ)-resolvent family $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$. Then, A generates the (α,β,μ)-resolvent sequence $\{S^n_{\alpha,\beta,\mu}\}_{n\in\mathbb{N}_0}$ given by $S^n_{\alpha,\beta,\mu} = \int_0^\infty \rho_n^\tau(t)S_{\alpha,\beta,\mu}(t)dt$.*

3. Construction of the method and existence of solutions.

Consider the problem

$${}_C\nabla^\alpha v^n + \mu\,{}_C\nabla^\beta v^n = Av^n + f^n, \quad n\geq 2. \tag{3.1}$$

where A is a sectorial operator and $(f^n)_{n\in\mathbb{N}_0}$ is a given sequence.

Since by definition, ${}_C\nabla^\alpha v^0 = {}_C\nabla^\alpha v^1 = 0$, for any $n \geq 2$, we get

$${}_C\nabla^\alpha v^n = \tau\sum_{j=2}^{n-1} k_\tau^{2-\alpha}(n-j)(\nabla_\tau^2 v)^j + \tau^{-\alpha}(v^n - 2v^{n-1} + v^{n-2}),$$

and, the same identity holds for β instead α. Then, (3.1) is equivalent to

$$(\tau^{-\alpha}+\tau^{-\beta}-A)v^n = 2(\tau^{-\alpha}+\tau^{-\beta})v^{n-1}-(\tau^{-\alpha}+\tau^{-\beta})v^{n-2} - \tau\sum_{j=2}^{n-1}[k_\tau^{2-\alpha}(n-j)-k_\tau^{2-\beta}(n-j)](\nabla_\tau^2 v)^j + f^n. \tag{3.2}$$

Consequently, to compute v^n for $n \geq 2$, it is necessary to know $v^{n-1}, v^{n-2}, ..., v^1, v^0$. To achieve this, we need to solve the equation (3.2) and we may define v^0 and v^1 as $u(0)$ and $u'(0)$, respectively (or their respective available approximations). Given that A is a sectorial operator, we can select a sufficiently small step size τ to ensure the invertibility of $(\tau^{-\alpha}+\tau^{-\beta}-A)$.

We conclude that if A is a sectorial operator and $\max\{\omega, 0\} < \tau^{-\alpha}+\tau^{-\beta}$, then the solution $(v^n)_{n\in\mathbb{N}_0}$ to (3.1) subject to the initial conditions $v^0 = u_0$ and $v^1 = u_1$ is given by

$$v^n = 2(\tau^{-\alpha}+\tau^{-\beta})(\tau^{-\alpha}+\tau^{-\beta}-A)^{-1}v^{n-1}-(\tau^{-\alpha}+\tau^{-\beta})(\tau^{-\alpha}+\tau^{-\beta}-A)^{-1}v^{n-2} - \tau\sum_{j=2}^{n-1}[k_\tau^{2-\alpha}(n-j)-k_\tau^{2-\beta}(n-j)](\tau^{-\alpha}+\tau^{-\beta}-A)^{-1}(\nabla_\tau^2 v)^j + (\tau^{-\alpha}+\tau^{-\beta}-A)^{-1}f^n, \quad n\geq 2. \tag{3.3}$$

Summarizing, we have the following result.

Proposition 3.15. *Let $A \in \mathrm{Sect}(\theta, \omega, M)$ in a Banach space X with $\max\{\omega, 0\} < \tau^{-\alpha}+\tau^{-\beta}$. Then, the solution $(v^n)_{n\in\mathbb{N}_0}$ to problem (3.1) is given by the sequence (3.3).*

Now, assume for the moment that $u : [0,\infty) \to X$ is a twice differentiable and bounded function. Suppose that A is the generator of an (α, β, μ)-resolvent family $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$.

Multiplying the equation (1.2) by $\rho_n^\tau(t)$ and integrating over $[0,\infty)$ we obtain, by Theorem 2.7, the discrete multi-term equation

$$_C\nabla^\alpha u^n + \mu\, _C\nabla^\beta u^n = Au^n + f^n, \quad n\geq 2, \tag{3.4}$$

where $u^n = \int_0^\infty \rho_n^\tau(t)u(t)dt$ and $f^n = \int_0^\infty \rho_n^\tau(t)f(t)dt$.

Take $u^0 := u_0$ and $u^1 := u_1$. Proceeding as above, we obtain that $(u^n)_{n\in\mathbb{N}}$ verifies the scheme

$$(\tau^{-\alpha}+\tau^{-\beta}-A)u^n = 2(\tau^{-\alpha}+\tau^{-\beta})u^{n-1}-(\tau^{-\alpha}+\tau^{-\beta})u^{n-2} - \tau\sum_{j=2}^{n-1}[k_\tau^{2-\alpha}(n-j)-k_\tau^{2-\beta}(n-j)](\nabla_\tau^2 u)^j + f^n. \tag{3.5}$$

We will now represent the solution to (3.5) using a variation-of-parameters formula involving the resolvent family $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$. Given the equivalence of (3.5) and (3.4), we apply the Z-transform to (3.4). Multiplying (3.4) by z^{-n} (where $|z| > 1$) and summing over $\mathbb{N}_0$ yields, according to Lemma 2.8, that

$$\left(\frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^\alpha + \mu\frac{1}{\tau^\beta}\left(\frac{z-1}{z}\right)^\beta - A\right)\tilde{u}(z) = \left(\frac{1}{\tau^\alpha}\left(\frac{z-1}{z}\right)^{\alpha-1} + \mu\frac{1}{\tau^\beta}\left(\frac{z-1}{z}\right)^{\beta-1}\right)u(0) + \left(\frac{1}{\tau^{\alpha-1}}\left(\frac{z-1}{z}\right)^{\alpha-2} + \mu\frac{1}{\tau^{\beta-1}}\left(\frac{z-1}{z}\right)^{\beta-2}\right)u'(0) + \tilde{f}(z).$$

As A generates the sequence $\{S^n_{\alpha,\beta,\mu}\}_{n\in\mathbb{N}_0}$ (see Proposition 2.14), by Proposition 2.13, we deduce that

$$\tilde{u}(z) = \tilde{S}_{\alpha,\beta,\mu}(z)u(0) + \mu\tau\tilde{k}_\tau^{\alpha-\beta}(z)\tilde{S}_{\alpha,\beta,\mu}(z)u(0) + \tau\tilde{k}_\tau^{1}(z)\tilde{S}_{\alpha,\beta,\mu}(z)u'(0) + \mu\tau\tilde{k}_\tau^{\alpha-\beta+1}(z)\tilde{S}_{\alpha,\beta,\mu}(z)u'(0) + \tau^2\tilde{k}_\tau^{\alpha-1}(z)\tilde{S}_{\alpha,\beta,\mu}(z)\tilde{f}(z).$$

Summarizing, we have proven the following result.

Proposition 3.16. *Let $\tau > 0$. Let A be the generator of a bounded (α, β, μ)-resolvent family $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$. If $u_0, u_1 \in X$ and f is bounded, then the fractional multi-term difference equation* (3.4) *has a unique solution given by*

$$u^n = S^n_{\alpha,\beta,\mu}u_0 + \mu\tau(k^{\alpha-\beta}_\tau \star S_{\alpha,\beta,\mu})^n u_0 + \tau(k^1_\tau \star S_{\alpha,\beta,\mu})^n u_1 + \mu\tau(k^{\alpha-\beta+1}_\tau \star S_{\alpha,\beta,\mu})^n u_1 + \tau^2(k^{\alpha-1}_\tau \star S_{\alpha,\beta,\mu} \star f)^n,$$

for all $n \geq 2$, and $u^0 = u(0)$, $u^1 = u'(0)$, where $S^n_{\alpha,\beta,\mu} := \int_0^\infty \rho^\tau_n(t) S_{\alpha,\beta,\mu}(t)dt$.

Now, given that $v^0 = u^0 = u_0$ and $v^1 = u^1 = u_1$, the sequences in (3.2) and (3.5) are identical. Consequently, without imposing any regularity on the sequence $(v^n)_n$, we have the following result.

Theorem 3.17. *Let $\tau > 0$. Let A be the generator of an (α, β, μ)-resolvent sequence $\{S^n_{\alpha,\beta,\mu}\}_{n\in\mathbb{N}_0}$. If $u_0, u_1 \in X$ and $(f^n)_{n\in\mathbb{N}_0}$ is a given sequence, then the fractional multi-term difference equation* (3.1) *has a unique solution given by*

(3.6)

$$v^n = S^n_{\alpha,\beta,\mu}u_0 + \mu\tau(k^{\alpha-\beta}_\tau \star S_{\alpha,\beta,\mu})^n u_0 + \tau(k^1_\tau \star S_{\alpha,\beta,\mu})^n u_1 + \mu\tau(k^{\alpha-\beta+1}_\tau \star S_{\alpha,\beta,\mu})^n u_1 + \tau^2(k^{\alpha-1}_\tau \star S_{\alpha,\beta,\mu} \star f)^n,$$

for all $n \geq 2$, and $v^0 = u_0$, $v^1 = u_0$.

From Proposition A.2, Proposition 2.14, and Theorem 3.17, we have the following Corollary.

Corollary 3.18. *Let $\mu \geq 0, 1 < \beta < \alpha \leq 2$ and $A \in \mathrm{Sect}(\theta, M)$ where $\theta = \frac{\alpha\pi}{2}$. If $\alpha - \beta \leq 1$, $u_0, u_1 \in X$ and (f^n) is a given sequence, then the fractional multi-term difference equation* (3.1) *has a unique solution given by* (3.6)*, where $\{S^n_{\alpha,\beta,\mu}\}_{n\in\mathbb{N}_0}$ is the (α, β, μ)-resolvent sequence generated by A.*

4. Convergence and error estimates for sectorial operators

In general, each term of the sequence v^n approximates the value of the function v at t_n, where $t_n = n\tau$ (for $\tau > 0$). In this section, we study the norm difference $\|u(t_n) - u^n\|$, where u is the mild solution to Problem (1.2) and u^n solves the discrete difference equation (3.4).

For a closed operator $A \in \mathrm{Sec}(\theta, M)$ and $t > 0$, we consider the path $\Gamma := \Gamma_t$ defined as: For $\frac{\pi}{2} < \theta < \pi$, we take ϕ such that $\frac{1}{2}\phi < \frac{\pi}{2}\alpha < \phi < \theta$. Next, we define Γ_t (see Figure 1) as the union $\Gamma^1_t \cup \Gamma^2_t$, where

$$\Gamma^1_t := \left\{ \frac{1}{t} e^{i\psi/\alpha} : -\phi < \psi < \phi \right\} \quad \text{and} \quad \Gamma^2_t := \left\{ re^{\pm i\phi/\alpha} : \frac{1}{t} \leq r \right\}.$$

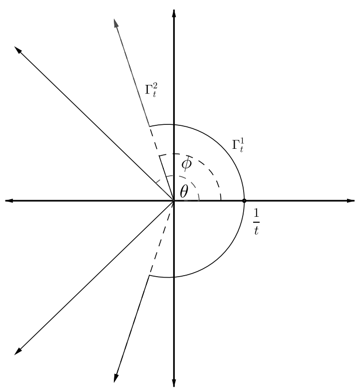


Figure 1. Plot of path Γ_t.

The next result will be useful to prove the main theorem in this section. For a similar result see [29].

Lemma 4.19. *Let $A \in \mathrm{Sec}(\theta, M)$ and Γ be the complex path defined above. If $\mu \geq 0$, then $\int_\Gamma \left|\frac{e^{zt}}{z^\mu}\right||dz| \leq C_\alpha t^{\mu-1}$ for all $t > 0$, where $C_\alpha := \left(2\phi \int_{-\phi}^{\phi} e^{\cos(\psi/\alpha)}d\psi + \frac{2}{-\cos(\phi/\alpha)}\right)$.*

If $A \in \mathrm{Sec}(\theta, M)$, then $z^\alpha + \mu z^\beta = h(z) \in \rho(A)$ (see Proposition A.2), and therefore, the inversion formula of the Laplace transform implies that

$$S_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_\Gamma e^{zt} z^{\alpha-1}(h(z) - A)^{-1}dz, \quad t > 0, \tag{4.1}$$

where $\Gamma := \Gamma_t$ is the path defined in Lemma 4.19.

Theorem 4.20. *Let $\mu \geq 0$ and $1 < \beta < \alpha \leq 2$ and $A \in \mathrm{Sect}(\theta, M)$ where $\theta = \frac{\alpha\pi}{2}$. Suppose that there exist $0 < \varepsilon_1 < 1$ such that $\alpha - \beta < \varepsilon_1$ and $1 + \varepsilon_1 < \alpha$. If there exists $K > 0$ such that $\|f(t)\| \leq K g_\gamma(t)$ for all $t \geq 0$, where $0 < \gamma < 1$, then the mild solution u to (1.2) satisfies $\|u(t)\| \to 0$ as $t \to \infty$.*

Proof. We know that the mild solution to (1.2) is given by

$$u(t) = S_{\alpha,\beta,\mu}(t)u_0 + \mu(g_{\alpha-\beta} * S_{\alpha,\beta,\mu})(t)u_0 + (g_1 * S_{\alpha,\beta,\mu})(t)u_1 + \mu(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)u_1 + (g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)(t).$$

By Theorem A.3 we have $\|S_{\alpha,\beta,\mu}(t)u_0\| \to 0$ as $t \to \infty$. Let $\Gamma := \Gamma_t$ be the path defined in Lemma 4.19. Now, as $\hat{g}_{\alpha-\beta}(z) = 1/z^{\alpha-\beta}$ and $A \in \mathrm{Sec}(\theta, M)$, we have

$$\|(g_{\alpha-\beta} * S_{\alpha,\beta,\mu})(t)\| \leq \frac{1}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|z|^{\alpha-\beta}}|z|^{\alpha-1}\|(h(z) - A)^{-1}\||dz| \leq \frac{M}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|z|^{\alpha-\beta}|z + \mu z^{\beta-\alpha+1} - \omega z^{1-\alpha}|}|dz|.$$

As $1 - \alpha < 0$ and $\beta - \alpha + 1 > 0$, we have $\frac{1}{|z+\mu z^{\beta-\alpha+1} - \omega z^{1-\alpha}|} \to 0$ as $|z| \to 0$ and $|z| \to \infty$. Therefore, there exists $\tilde{M}_1 > 0$ such that $\frac{1}{|z+\mu z^{\beta-\alpha+1} - \omega z^{1-\alpha}|} \leq \tilde{M}_1$ for all z such that $h(z) \in \rho(A)$. Since $\alpha - \beta < \varepsilon_1$, the Lemma 4.19 implies that

$$\|(g_{\alpha-\beta} * S_{\alpha,\beta,\mu})(t)\| \leq \frac{M\tilde{M}_1}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|z|^{\alpha-\beta}}|dz| \leq \frac{C_\alpha M\tilde{M}_1}{2\pi}t^{\alpha-\beta-1} \to 0 \quad \text{as} \quad t \to \infty.$$

Similarly,

$$\|(g_1 * S_{\alpha,\beta,\mu})(t)\| \leq \frac{M}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|z|^{1-\varepsilon_1}}\frac{|z|^{\alpha-1-\varepsilon_1}}{|z^\alpha + \mu z^\beta - \omega|}|dz|.$$

Since $\alpha - \beta < \varepsilon_1$ and $1 + \varepsilon_1 < \alpha$ we have $\frac{|z|^{\alpha-1-\varepsilon_1}}{|z^\alpha+\mu z^\beta-\omega|} = \frac{1}{|z^{1+\varepsilon_1}+\mu z^{\beta-\alpha+1+\varepsilon_1} - \omega z^{1+\varepsilon_1-\alpha}|} \to 0$ as $|z| \to 0$ and $|z| \to \infty$. Thus, there exists $\tilde{M}_2 > 0$ such that $\frac{1}{|z^{1+\varepsilon_1}+\mu z^{\beta-\alpha+1+\varepsilon_1} - \omega z^{1+\varepsilon_1-\alpha}|} \leq \tilde{M}_2$ for all z with $h(z) \in \rho(A)$. By Lemma 4.19 we obtain

$$\|(g_1 * S_{\alpha,\beta,\mu})(t)\| \leq \frac{C_\alpha M\tilde{M}_2}{2\pi}t^{-\varepsilon_1} \to 0 \quad \text{as} \quad t \to \infty.$$

Analogously, we obtain

$$\|(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)\| \leq \frac{C_\alpha M\tilde{M}_1}{2\pi}t^{\alpha-\beta-\varepsilon_1} \to 0 \quad \text{as} \quad t \to \infty.$$

Finally, since $\|f(t)\| \leq K g_\gamma(t)$ for any $t > 0$, we have $\|\widehat{(g_{\alpha-1} * f)}(z)\| \leq \frac{K}{|z|^{\alpha+\gamma-1}}$ for any $\mathrm{Re}(z) > 0$. Then

$$\begin{aligned}
\|(g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)(t)\| &\leq \frac{1}{2\pi}\int_\Gamma |e^{zt}|\|S_{\alpha,\beta,\mu}(t)\|\|\widehat{(g_{\alpha-1} * f)}(z)\||dz| \\
&\leq \frac{K}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|z|^{\alpha+\gamma-1}}|z|^{\alpha-1}\|(h(z) - A)^{-1}\||dz| \\
&\leq \frac{MK}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|z|^\gamma}\frac{1}{|z^\alpha + \mu z^\beta - \omega|}|dz|.
\end{aligned}$$

As $\alpha, \beta > 0$ we have $\frac{1}{|z^\alpha+\mu z^\beta-\omega|} \to 0$ as $|z| \to \infty$ and $\frac{1}{|z^\alpha+\mu z^\beta-\omega|} \to \frac{1}{|\omega|}$ as $|z| \to 0$. Therefore, there exists $\tilde{M}_3 > 0$ such that $\frac{1}{|z^\alpha+\mu z^\beta-\omega|} \le \tilde{M}_3$. By Lemma 4.19 we conclude that

$$\|(g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)(t)\| \le \frac{C_\alpha M\tilde{M}_3 K}{2\pi} t^{\gamma-1} \to 0 \quad \text{as} \quad t \to \infty.$$

□

For a given $0 < \varepsilon < 1$, the space of all continuous function $f : [0,\infty) \to D(A^\varepsilon)$ endowed with the norm $\|f\|_\varepsilon := \sup_{t\ge 0} \|f(t)\|_\varepsilon = \sup_{t\ge 0} \|A^\varepsilon f(t)\|$ will be denoted by $C([0,\infty), D(A^\varepsilon))$.

Theorem 4.21. *Let $\mu > 0$ and $1 < \beta < \alpha \le 2$ and $A \in \mathrm{Sect}(\theta, M)$ where $\theta = \frac{\alpha\pi}{2}$. Let $0 < \varepsilon < 1$ such that $1 < \beta(\varepsilon + 1) < \alpha$ and $0 < \beta\varepsilon < 1$. Suppose that $f \in C([0,\infty), D(A^\varepsilon))$. Let Γ be the complex path defined above. If $u_0, u_1 \in D(A^\varepsilon)$, then for each $T > 0$ there exists a constant $C = C(T) > 0$ (independent of the solution, the data and the step size) such that, for $0 < t_n \le T$, there holds*

$$\|u^n - u(t_n)\| \le C\tau t_n^{\beta\varepsilon-1} \left(\|u_0\|_\varepsilon + \|u_1\|_\varepsilon + \|f\|_\varepsilon\right).$$

Proof. By Proposition A.2, the operator $A \in \mathrm{Sec}(\theta, M)$ generates an (α,β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t\ge 0}$. The solution to (1.2) is given by

$$u(t) = S_{\alpha,\beta,\mu}(t)u_0 + \mu(g_{\alpha-\beta} * S_{\alpha,\beta,\mu})(t)u_0 + (g_1 * S_{\alpha,\beta,\mu})(t)u_1 + \mu(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)u_1 + (g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)(t),$$

and by Theorem 3.17, the solution to the equation (3.4) is given by

$$u^n = S^n_{\alpha,\beta,\mu}u_0 + \mu\tau(g_{\alpha-\beta} \star S_{\alpha,\beta,\mu})^n u_0 + \tau(g_1 \star S_{\alpha,\beta,\mu})^n u_1 + \mu\tau(g_{\alpha-\beta+1} \star S_{\alpha,\beta,\mu})^n u_1 + \tau^2(g_{\alpha-1} \star S_{\alpha,\beta,\mu} \star f)^n,$$

where $S^n_{\alpha,\beta,\mu} = \int_0^\infty \rho^\tau_n(t) S_{\alpha,\beta,\mu}(t)dt$. Fix $n \in \mathbb{N}$ such that $0 < t_n \le T$, where $t_n := \tau n$. Then, we have

$$\begin{aligned}\|u^n - u(t_n)\| \le\ & \|(S_{\alpha,\beta,\mu}(t_n) - S^n_{\alpha,\beta,\mu})u_0\| + \mu\|((g_{\alpha-\beta} * S_{\alpha,\beta,\mu})(t_n) - \tau(k^{\alpha-\beta}_\tau \star S_{\alpha,\beta,\mu})^n)u_0\| \\ & + \|((g_1 * S_{\alpha,\beta,\mu})(t_n) - \tau(k^1_\tau \star S_{\alpha,\beta,\mu})^n)u_1\| \\ & + \mu\|((g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t_n) - \tau(k^{\alpha-\beta+1}_\tau \star S_{\alpha,\beta,\mu})^n)u_1\| \\ & + \|(g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)(t_n) - \tau^2(k^{\alpha-1}_\tau \star S_{\alpha,\beta,\mu} \star f)^n\| := I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned}$$

Now, we estimate each term I_j for $j = 1, 2, ..., 5$. Since $\int_0^\infty \rho^\tau_n(t)dt = 1$, we can write

$$(S_{\alpha,\beta,\mu}(t_n) - S^n_{\alpha,\beta,\mu})u_0 = \int_0^\infty \rho^\tau_n(t)((S_{\alpha,\beta,\mu}(t_n) - S_{\alpha,\beta,\mu}(t))u_0 dt,$$

and therefore $I_1 \le \int_0^\infty \rho^\tau_n(t)\|(S_{\alpha,\beta,\mu}(t) - S_{\alpha,\beta,\mu}(t_n))u_0\|dt$. Now, by (4.1) we can write

$$(S_{\alpha,\beta,\mu}(t) - S_{\alpha,\beta,\mu}(t_n))u_0 = \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt} - e^{zt_n})}{z} z^\alpha (h(z) - A)^{-1}u_0 dz,$$

where $h(z) = z^\alpha + \mu z^\beta$. Since $A(h(z) - A)^{-1} = A^{1-\varepsilon}(h(z) - A)^{-1}A^\varepsilon$ we have

$$h(z)(h(z) - A)^{-1} = A(h(z) - A)^{-1} + I = A^{1-\varepsilon}(h(z) - A)^{-1}A^\varepsilon + I. \tag{4.2}$$

Moreover, we can write

$$z^\alpha(h(z)-A)^{-1} = h(z)(h(z)-A)^{-1} - \mu z^\beta(h(z)-A)^{-1} = h(z)(h(z)-A)^{-1} - \mu\frac{z^\beta}{h(z)}h(z)(h(z)-A)^{-1}, \tag{4.3}$$

and therefore

$$\begin{aligned}(S_{\alpha,\beta,\mu}(t) - S_{\alpha,\beta,\mu}(t_n))u_0 =\ & \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt} - e^{zt_n})}{z}\left(1 - \mu\frac{z^\beta}{h(z)}\right)u_0 dz \\ & + \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt} - e^{zt_n})}{z}\left(1 - \mu\frac{z^\beta}{h(z)}\right)A^{1-\varepsilon}(h(z) - A)^{-1}A^\varepsilon u_0 dz.\end{aligned}$$

Since $p(z) := \frac{(e^{zt}-e^{zt_n})}{z}$ and $q(z) := \mu p(z)\frac{z^\beta}{h(z)} = \mu\frac{(e^{zt}-e^{zt_n})}{z}\cdot\frac{1}{z^{\alpha-\beta}+\mu}$ have a unique removable singularity at $z=0$ and $t\geq t_n$ we obtain that they can be analytically extended to the region enclosed by the path $\Gamma^R := \Gamma^R_{t_n}$ where Γ^R is the path given in Figure 2, and therefore $\frac{1}{2\pi i}\int_{\Gamma^R}\frac{(e^{zt}-e^{zt_n})}{z}u^0dz = 0$. Since $\frac{1}{2\pi i}\int_{\Gamma}\frac{(e^{zt}-e^{zt_n})}{z}u^0dz = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma^R}\frac{(e^{zt}-e^{zt_n})}{z}u^0dz$, we obtain $\frac{1}{2\pi i}\int_{\Gamma}\frac{(e^{zt}-e^{zt_n})}{z}u^0dz = 0$. Similar result holds for $q(z)$ and therefore

$$\frac{1}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zt_n})}{z}u_0dz = \frac{\mu}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zt_n})}{z}\frac{z^\beta}{h(z)}u_0dz = 0.$$

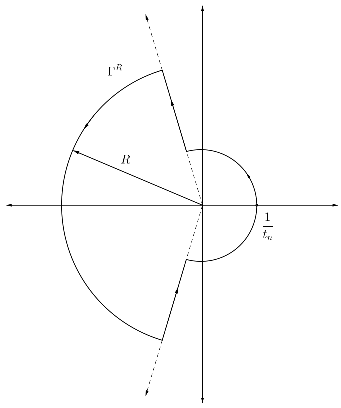


FIGURE 2. Plot of path Γ^R.

On the other hand, since A is a sectorial operator, we get by (2.3)

$$\|A^{1-\varepsilon}(h(z)-A)^{-1}A^\varepsilon x\| \leq \kappa(M+1)\frac{\|A^\varepsilon x\|}{|h(z)|^\varepsilon}, \tag{4.4}$$

for all $x\in D(A^\varepsilon)$. Therefore,

$$\|(S_{\alpha,\beta,\mu}(t)-S_{\alpha,\beta,\mu}(t_n))u_0\| \leq \frac{\kappa(M+1)}{2\pi}\left(\int_\Gamma\frac{|e^{zt}-e^{zt_n}|}{|z|}\frac{1}{|h(z)|^\varepsilon}|dz| + \int_\Gamma\frac{|e^{zt}-e^{zt_n}|}{|z|}\frac{\mu|z|^\beta}{|h(z)|^{\varepsilon+1}}|dz|\right)\|A^\varepsilon u_0\|.$$

We notice that $\frac{1}{h(z)} = \frac{1}{z^\beta}\cdot\frac{1}{z^{\alpha-\beta}+\mu}$. Now, we write $z^{\alpha-\beta} = re^{i\phi}$. If $\mathrm{Re}(z^{\alpha-\beta})\geq 0$, then $\frac{1}{|z^{\alpha-\beta}+\mu|}\leq\frac{2}{\mu}$. Now, if $\mathrm{Re}(z^{\alpha-\beta})<0$, then $\cos(\phi)<0$ and $\frac{1}{|z^{\alpha-\beta}+\mu|}\leq\frac{r+\mu}{r^2+2\mu r\cos(\phi)+\mu^2} =: f(r)$. An easy computation shows that $\lim_{r\to 0}f(r)=\frac{1}{\mu}$, $\lim_{r\to\infty}f(r)=0$, and that $f(r)$ has a maximum at $r_0 := -\mu+2\sqrt{2}\mu\sqrt{1-\cos(\phi)}$. Thus

$$f(r)\leq f(r_0) = \frac{2\sqrt{2}}{\mu}\frac{1}{\sqrt{1-\cos(\phi)}(\sqrt{1-\cos(\phi)}-2\sqrt{2})^2} =: \frac{2\sqrt{2}}{\mu}h_1(\phi),$$

for all $r\geq 0$. Since $\cos(\phi)<0$ we may assume that $\pi/2<\phi<\pi$, which implies that $h_1(\phi)\leq 1/2$ and therefore $f(r)\leq\frac{\sqrt{2}}{\mu}$. We conclude that $\frac{1}{|z^{\alpha-\beta}+\mu|}\leq\max\left\{\frac{2}{\mu},\frac{\sqrt{2}}{\mu}\right\}=\frac{2}{\mu}$, which implies that

$$\frac{1}{|h(z)|}\leq\frac{2}{\mu|z|^\beta}, \tag{4.5}$$

for all $\mathrm{Re}z>0$. Moreover, by the generalized mean value theorem, there exist t_0, t_1 with $0<t_n<t_0<t_1<t$ such that

$$\frac{|e^{zt}-e^{zt_n}|}{|z|}\leq (t-t_n)\left(|e^{t_0z}|+|e^{t_1z}|\right), \tag{4.6}$$

and by Lemma 4.19 and (4.5) we obtain

$$\int_\Gamma \frac{|e^{zt}-e^{zt_n}|}{|z|}\frac{1}{|h(z)|^\varepsilon}|dz| \le \left(\frac{2}{\mu}\right)^\varepsilon (t-t_n)\int_\Gamma \frac{|e^{t_0z}|+|e^{t_1z}|}{|z|^{\beta\varepsilon}}|dz| \le \left(\frac{2}{\mu}\right)^\varepsilon (t-t_n)C_\alpha(t_0^{\beta\varepsilon-1}+t_1^{\beta\varepsilon-1}),$$

and

$$\int_\Gamma \frac{|e^{zt}-e^{zt_n}|}{|z|}\frac{\mu|z|^\beta}{|h(z)|^{\varepsilon+1}}|dz| \le \mu\left(\frac{2}{\mu}\right)^{1+\varepsilon} (t-t_n)\int_\Gamma \frac{|e^{t_0z}|+|e^{t_1z}|}{|z|^{\beta\varepsilon}}|dz| \le \mu\left(\frac{2}{\mu}\right)^{\varepsilon+1} (t-t_n)C_\alpha(t_0^{\beta\varepsilon-1}+t_1^{\beta\varepsilon-1}).$$

Therefore, we have that

$$\|(S_{\alpha,\beta,\mu}(t)-S_{\alpha,\beta,\mu}(t_n))u_0\| \le \frac{3\kappa(M+1)}{2\pi}\left(\frac{2}{\mu}\right)^\varepsilon (t-t_n)C_\alpha(t_0^{\beta\varepsilon-1}+t_1^{\beta\varepsilon-1})\|A^\varepsilon u_0\|.$$

Since $0<\beta\varepsilon<1$ and $t_n<t_0<t_1$ we obtain $t_1^{\beta\varepsilon-1}<t_0^{\beta\varepsilon-1}<t_n^{\beta\varepsilon-1}$ and thus

$$\|(S_{\alpha,\beta,\mu}(t)-S_{\alpha,\beta,\mu}(t_n))u_0\| \le \frac{3\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^\varepsilon (t-t_n)C_\alpha t_n^{\beta\varepsilon-1}\|A^\varepsilon u_0\| =: D_1(t-t_n)t_n^{\beta\varepsilon-1}\|A^\varepsilon u_0\|.$$

Since $\int_0^\infty \rho_n^\tau(t)dt=1$ for all $n\in\mathbb{N}$, we have

$$\int_0^\infty \rho_n^\tau(t)(t-t_n)dt = \int_0^\infty \rho_n^\tau(t)tdt - t_n = t_{n+1}-t_n=\tau, \tag{4.7}$$

and we conclude that

$$\int_0^\infty \rho_n^\tau(t)\|(S_{\alpha,\beta,\mu}(t)-S_{\alpha,\beta,\mu}(t_n))u_0\|dt \le D_1t_n^{\beta\varepsilon-1}\|A^\varepsilon u_0\|\int_0^\infty \rho_n^\tau(t)(t-t_n)dt \le D_1\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_0\|,$$

for all $n\in\mathbb{N}$, and thus

$$I_1 \le D_1\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_0\|.$$

To estimate I_2 we notice that, by Theorem 2.9, I_2 can be written as

$$I_2 = \mu\left\|\int_0^\infty \rho_n^\tau(t)[(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})(t_n)-(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})(t)]u_0dt\right\|. \tag{4.8}$$

Since $\widehat{(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})}(z)=\frac{1}{z^{\alpha-\beta}}\hat{S}_{\alpha,\beta,\mu}(z)=\frac{1}{z}\frac{z^\beta}{h(z)}h(z)(h(z)-A)^{-1}$, for all $\mathrm{Re}(z)>0$, and by (4.2) and (4.3) we can write $\widehat{(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})}(z)=\frac{1}{z}\frac{z^\beta}{h(z)}A^{1-\varepsilon}(h(z)-A)^{-1}A^\varepsilon+\frac{1}{z}\frac{z^\beta}{h(z)}I$. By the inversion theorem for the Laplace transform, we have

$$(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})(t)u_0=\frac{1}{2\pi i}\int_\Gamma e^{zt}\widehat{(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})}(z)u_0dz,$$

and therefore, for $u_0\in D(A^\varepsilon)$, we have

$$\begin{aligned}(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})(t_n)u_0-(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})(t)u_0 \;&=\; \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt_n}-e^{zt})}{z}\frac{z^\beta}{h(z)}A^{1-\varepsilon}(h(z)-A)^{-1}A^\varepsilon u_0dz\\ &\quad+\frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt_n}-e^{zt})}{z}\frac{z^\beta}{h(z)}u_0dz.\end{aligned}$$

The second integral in this last equality is equal to zero, because $p(z):=\frac{(e^{zt}-e^{zt_n})}{z}$ and $q(z):=p(z)\frac{z^\beta}{h(z)}$ have a unique removable singularity at $z=0$. By the inequality (4.4) we have

$$\|[(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})(t_n)-(g_{\alpha-\beta}*S_{\alpha,\beta,\mu})(t)]u_0\| \;\le\; \frac{\kappa(M+1)}{2\pi}\int_\Gamma \frac{|e^{zt_n}-e^{zt}|}{|z|}\frac{|z|^\beta}{|h(z)|^{\varepsilon+1}}\|A^\varepsilon u_0\||dz|.$$

By Lemma 4.19 and (4.5)-(4.6) we have

$$\int_\Gamma \frac{|e^{zt}-e^{zt_n}|}{|z|}\frac{|z|^\beta}{|h(z)|^{\varepsilon+1}}|dz| \le \left(\frac{2}{\mu}\right)^{1+\varepsilon}(t-t_n)\int_\Gamma \frac{|e^{t_0z}|+|e^{t_1z}|}{|z|^{\beta\varepsilon}}|dz| \le \left(\frac{2}{\mu}\right)^{\varepsilon+1}(t-t_n)C_\alpha(t_0^{\beta\varepsilon-1}+t_1^{\beta\varepsilon-1}).$$

Since $\beta\varepsilon - 1 < 0$ and $0 < t_n < t_0 < t_1$ we have $t_1^{\beta\varepsilon-1} < t_0^{\beta\varepsilon-1} < t_n^{\beta\varepsilon-1}$ and we get

$$\|[(g_{\alpha-\beta} * S_{\alpha,\beta,\mu})(t_n) - (g_{\alpha-\beta} * S_{\alpha,\beta,\mu})(t)]u_0\| \leq \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha(t-t_n)t_n^{\beta\varepsilon-1}\|A^\varepsilon u_0\|.$$

Therefore, by (4.7) and (4.8) we have

$$I_2 \leq D_2\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_0\|,$$

where $D_2 := \frac{\mu\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha$.

Next, we estimate I_3. Since

$$[(g_1 * S_{\alpha,\beta,\mu})(t_n) - \tau(k_\tau^1 \star S_{\alpha,\beta,\mu})^n(t)]u_1 = \int_0^\infty \rho_n^\tau(t)[(g_1 * S_{\alpha,\beta,\mu})(t_n) - (g_1 * S_{\alpha,\beta,\mu})(t)]u_1 dt,$$

and $(\widehat{g_1 * S_{\alpha,\beta,\mu}})(z) = \frac{1}{z}\hat{S}_{\alpha,\beta,\mu}(z) = \frac{1}{z}\frac{z^{\alpha-1}}{h(z)}h(z)(h(z)-A)^{-1}$, for all $\mathrm{Re}(z) > 0$, we have by (4.2) that

$$\begin{aligned}[(g_1 * S_{\alpha,\beta,\mu})(t_n) - (g_1 * S_{\alpha,\beta,\mu})(t)]u_1 &= \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt_n} - e^{zt})}{z}\frac{z^{\alpha-1}}{h(z)}A^{1-\varepsilon}(h(z)-A)^{-1}A^\varepsilon u_1 dz \\ &\quad + \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt_n} - e^{zt})}{z}\frac{z^{\alpha-1}}{h(z)}u_1 dz \\ &=: J_1 + J_2.\end{aligned}$$

Since $q(z) := \frac{(e^{zt_n} - e^{zt})}{z} \cdot \frac{z^{\alpha-1}}{h(z)} = \frac{(e^{zt_n} - e^{zt})}{z^{2+\beta-\alpha}} \cdot \frac{1}{z^{\alpha-\beta}+\mu}$ has a unique removable singularity at $z = 0$, the integral J_2 is equal to zero.

On the other hand, by (4.4), (4.5) and (4.6) we have

$$\begin{aligned}\|J_1\| &\leq \frac{\kappa(M+1)}{2\pi}\int_\Gamma \frac{|e^{zt_n} - e^{zt}|}{|z|}\frac{|z|^{\alpha-1}}{|h(z)|^{\varepsilon+1}}\|A^\varepsilon u_1\||dz| \\ &\leq \frac{\kappa(M+1)}{2\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1}(t-t_n)\int_\Gamma (|e^{zt_0}| + |e^{zt_1}|)\frac{1}{z^{\beta(\varepsilon+1)-\alpha+1}}\|A^\varepsilon u_1\||dz|.\end{aligned}$$

Since $1 < \beta(\varepsilon+1) < \alpha$ and $\alpha > 1$, we obtain $0 < \beta(\varepsilon+1) - \alpha + 1 < 1$, and the Lemma 4.19 implies that

$$\|J_1\| \leq \frac{\kappa(M+1)}{2\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha(t-t_n)(t_0^{\beta(\varepsilon+1)-\alpha} + t_1^{\beta(\varepsilon+1)-\alpha})\|A^\varepsilon u_1\|.$$

The condition $1 < \beta(\varepsilon+1) < \alpha$ implies that $t_1^{\beta(\varepsilon+1)-\alpha} < t_0^{\beta(\varepsilon+1)-\alpha} < t_n^{\beta(\varepsilon+1)-\alpha}$ and thus

$$\begin{aligned}\|J_1\| &\leq \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha(t-t_n)t_n^{\beta(\varepsilon+1)-\alpha}\|A^\varepsilon u_1\| \\ &\leq \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha T^{\beta-\alpha+1}(t-t_n)t_n^{\beta\varepsilon-1}\|A^\varepsilon u_1\|,\end{aligned}$$

because $\beta - \alpha + 1 > 0$. By (4.7) we conclude that

$$\begin{aligned}I_3 &\leq \int_0^\infty \rho_n^\tau(t)\|(g_1 * S_{\alpha,\beta,\mu})(t_n) - (g_1 * S_{\alpha,\beta,\mu})(t)\|u_1 dt \\ &\leq \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha T^{\beta-\alpha+1}\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_1\| \\ &= D_3\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_1\|,\end{aligned}$$

where $D_3 := \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha T^{\beta-\alpha+1}$.

Now, we estimate I_4. Since $(\widehat{g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu}})(z) = \frac{1}{z^{\alpha-\beta+1}}\hat{S}_{\alpha,\beta,\mu}(z) = z^{\beta-2}(h(z)-A)^{-1}$, for all $\mathrm{Re}(z) > 0$, by Theorem 2.9 we have

$$I_4 \leq \mu \int_0^\infty \rho_n^\tau(t)\|[(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t_n) - (g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)]u_1\|dt.$$

By the inversion theorem for the Laplace transform and (4.2) we get

$$\begin{aligned}[(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t_n) - (g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)]u_1 &= \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt_n} - e^{zt})}{z}\frac{z^{\beta-1}}{h(z)}A^{1-\varepsilon}(h(z)-A)^{-1}A^\varepsilon u_1 dz \\ &+ \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt_n} - e^{zt})}{z}\frac{z^{\beta-1}}{h(z)}u_1 dz.\end{aligned}$$

The inequalities (4.4), (4.5) and (4.6) imply that

$$\begin{aligned}\|[(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t_n) - (g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)]u_1\| &\leq \frac{\kappa(M+1)}{2\pi}\int_\Gamma \frac{|e^{zt_n} - e^{zt}|}{|z|}\frac{|z|^{\beta-1}}{|h(z)|^{\varepsilon+1}}\|A^\varepsilon u_1\||dz| \\ &+ \frac{1}{2\pi}\int_\Gamma \frac{|e^{zt_n} - e^{zt}|}{|z|}\frac{|z|^{\beta-1}}{|h(z)|}\|u_1\||dz| \\ &\leq \frac{\kappa(M+1)}{2\pi}(t-t_n)\left(\frac{2}{\mu}\right)^{\varepsilon+1}\int_\Gamma \frac{|e^{zt_0}| + |e^{zt_1}|}{|z|^{\beta\varepsilon+1}}\|A^\varepsilon u_1\||dz| \\ &+ \frac{1}{2\pi}\left(\frac{2}{\mu}\right)(t-t_n)\int_\Gamma (|e^{zt_0}| + |e^{zt_1}|)\frac{1}{|z|}\|u_1\||dz|.\end{aligned}$$

Since $\|u_1\| \leq \|A^\varepsilon u_1\|$, the Lemma 4.19 implies that

$$\begin{aligned}\|[(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t_n) - (g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)]u_1\| &\leq \frac{\kappa(M+1)}{2\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1}(t-t_n)C_\alpha(t_0^{\beta\varepsilon} + t_1^{\beta\varepsilon})\|A^\varepsilon u_1\| \\ &+ \frac{1}{\pi}\left(\frac{2}{\mu}\right)(t-t_n)C_\alpha\|A^\varepsilon u_1\|.\end{aligned}$$

Moreover, since $\beta\varepsilon > 0$ and $t_0 < t_1 < t$ we get $t_0^{\beta\varepsilon} < t^{\beta\varepsilon}$ and $t_1^{\beta\varepsilon} < t^{\beta\varepsilon}$, which implies that

$$\begin{aligned}\|[(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t_n) - (g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)]u_1\| &\leq \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1}(t-t_n)C_\alpha t^{\beta\varepsilon}\|A^\varepsilon u_1\| \\ &+ \frac{1}{\pi}\left(\frac{2}{\mu}\right)(t-t_n)C_\alpha\|A^\varepsilon u_1\|.\end{aligned}$$

Therefore,

$$\begin{aligned}I_4 &\leq \mu\int_0^\infty \rho_n^\tau(t)\|[(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t_n) - (g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)]u_1\|dt \\ &\leq \frac{\mu\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1}C_\alpha\int_0^\infty \rho_n^\tau(t)(t-t_n)t^{\beta\varepsilon}\|A^\varepsilon u_1\|dt + \frac{2}{\pi}C_\alpha\int_0^\infty \rho_n^\tau(t)(t-t_n)\|A^\varepsilon u_1\|dt.\end{aligned}$$

Now, an easy computation shows that for all $\eta > 0$

$$\int_0^\infty \rho_n^\tau(t)t^\eta dt = \frac{\tau^\eta}{n!}\Gamma(n+\eta+1), \tag{4.9}$$

for all $n \in \mathbb{N}$, and therefore

$$\int_0^\infty \rho_n^\tau(t)(t-t_n)t^\eta dt = \frac{\tau^{\eta+1}}{n!}\Gamma(n+\eta+2) - \frac{\tau^\eta}{n!}\Gamma(n+\eta+1)t_n =: c_n^\eta.$$

Now, c_n^η can be written as

$$\frac{\tau^{\eta+1}}{n!}\Gamma(n+\eta+2)-\frac{\tau^\eta}{n!}\Gamma(n+\eta+1)t_n \quad = \quad \tau(\eta+1)(n+\eta)t_n^\eta\frac{\Gamma(n+\eta)}{\Gamma(n+1)}\frac{1}{n^\eta}.$$

Since $\frac{\Gamma(n+\eta)}{\Gamma(n+1)} < n^{\eta-1}$ for all $0<\eta<1$ and $n\in\mathbb{N}_0$ (see for instance [10]), we have

$$c_n^\eta < \tau(\eta+1)(n+\eta)t_n^\eta n^{\eta-1}\frac{1}{n^\eta} = \tau(\eta+1)t_n^\eta\left(1+\frac{\eta}{n}\right) \le \tau(\eta+1)^2t_n^\eta.$$

for all $n\in\mathbb{N}$. If $\eta=\beta\varepsilon$, then the hypothesis implies that $c_n^{\beta\varepsilon} \le \tau(\beta\varepsilon+1)^2t_n^{\beta\varepsilon} = \tau t_n(\beta\varepsilon+1)^2t_n^{\beta\varepsilon-1} \le \tau(\beta\varepsilon+1)^2Tt_n^{\beta\varepsilon-1}$. This last inequality and (4.7) imply that

$$\begin{aligned} I_4 &\le \frac{\mu\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1}C_\alpha c_n^{\beta\varepsilon}\|A^\varepsilon u_1\| + \frac{2}{\pi}C_\alpha\tau\|A^\varepsilon u_1\| \\ &\le \frac{\mu\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1}C_\alpha(\beta\varepsilon+1)^2T\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_1\| + \frac{2}{\pi}C_\alpha T^{1-\beta\varepsilon}\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_1\|. \end{aligned}$$

We conclude that

$$I_4 \le D_4\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_1\|,$$

where the constant D_4 is defined by $D_4 := \left(\frac{\mu\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1}C_\alpha(\beta\varepsilon+1)^2T + \frac{2}{\pi}C_\alpha T^{1-\beta\varepsilon}\right)$.

Finally, we estimate I_5. By [23, Lemma 2.7] we can write

$$I_5 = \left\|\int_0^\infty \rho_n^\tau(t)[(g_{\alpha-1}*S_{\alpha,\beta,\mu}*f)(t)-(g_{\alpha-1}*S_{\alpha,\beta,\mu}*f)(t_n)]dt\right\|.$$

Moreover, we have

$$\begin{aligned} (g_{\alpha-1}*S_{\alpha,\beta,\mu}*f)(t)-(g_{\alpha-1}*S_{\alpha,\beta,\mu}*f)(t_n) &= \int_0^{t_n}[(g_{\alpha-1}*S_{\alpha,\beta,\mu})(t-r)-(g_{\alpha-1}*S_{\alpha,\beta,\mu})(t_n-r)]f(r)dr \\ &+ \int_{t_n}^t (g_{\alpha-1}*S_{\alpha,\beta,\mu})(t-r)f(r)dr \\ &:= J_1+J_2. \end{aligned}$$

In order to estimate J_1 we observe that $\widehat{(g_{\alpha-1}*S_{\alpha,\beta,\mu})}(z) = \frac{1}{z^{\alpha-1}}\hat{S}_{\alpha,\beta,\mu}(z) = (h(z)-A)^{-1}$, for all $\mathrm{Re}(z)>0$, which implies by (4.2) that

$$\begin{aligned} (g_{\alpha-1}*S_{\alpha,\beta,\mu})(t)x-(g_{\alpha-1}*S_{\alpha,\beta,\mu})(s)x &= \frac{1}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zs})}{h(z)}A^{1-\varepsilon}(h(z)-A)^{-1}A^\varepsilon x dz \\ &+ \frac{1}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zs})}{h(z)}x dz, \end{aligned}$$

for all $x\in D(A^\varepsilon)$ and $t>s>0$. By (4.4) and (4.5) we obtain

$$\begin{aligned} \|(g_{\alpha-1}*S_{\alpha,\beta,\mu})(t)x-(g_{\alpha-1}*S_{\alpha,\beta,\mu})(s)x\| &\le \frac{\kappa(M+1)}{2\pi}\int_\Gamma\frac{|e^{zt}-e^{zs}|}{|h(z)|^{\varepsilon+1}}\|A^\varepsilon x\||dz| \\ &+ \frac{1}{2\pi}\int_\Gamma\frac{|e^{zt}-e^{zs}|}{|h(z)|}\|A^\varepsilon x\||dz| \\ &\le \frac{\kappa(M+1)}{2\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1}\int_\Gamma\frac{|e^{zt}-e^{zs}|}{|z|}\frac{1}{|z|^{\beta(\varepsilon+1)-1}}\|A^\varepsilon x\||dz| \\ &+ \frac{1}{2\pi}\left(\frac{2}{\mu}\right)\int_\Gamma\frac{|e^{zt}-e^{zs}|}{|z|}\frac{1}{|z|^{\beta-1}}\|A^\varepsilon x\||dz|. \end{aligned}$$

The generalized mean value implies the existence of t_0, t_1 with $0 < s < t_0 < t_1 < t$ such that $\frac{|e^{zt}-e^{zs}|}{|z|} \leq (t-s)\left(|e^{t_0 z}| + |e^{t_1 z}|\right)$. Hence, by Lemma 4.19 we get

$$\|(g_{\alpha-1} * S_{\alpha,\beta,\mu})(t)x - (g_{\alpha-1} * S_{\alpha,\beta,\mu})(s)x\| \leq \frac{\kappa(M+1)}{2\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha (t-s)(t_0^{\beta(\varepsilon+1)-2} + t_1^{\beta(\varepsilon+1)-2})\|A^\varepsilon x\| + \frac{1}{\mu\pi} C_\alpha (t-s)(t_0^{\beta-2} + t_1^{\beta-2})\|A^\varepsilon x\|.$$

Since $1 < \beta(\varepsilon+1) < \alpha$, $\beta > 1$, and $0 < s < t_0 < t_1 < t$ we obtain

$$\begin{aligned} \|(g_{\alpha-1} * S_{\alpha,\beta,\mu})(t)x - (g_{\alpha-1} * S_{\alpha,\beta,\mu})(s)x\| &\leq \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha (t-s) s^{\beta(\varepsilon+1)-2}\|A^\varepsilon x\| \\ &+ \frac{2}{\mu\pi} C_\alpha (t-s) s^{\beta-2}\|A^\varepsilon x\|. \end{aligned}$$

Replacing t by $t-r$ and s by $t_n - r$ we obtain

$$\begin{aligned} \|J_1\| &\leq \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha (t-t_n)\int_0^{t_n} (t_n-r)^{\beta(\varepsilon+1)-2}\|A^\varepsilon f(r)\|dr \\ &+ \frac{2}{\mu\pi} C_\alpha (t-t_n)\int_0^{t_n}(t_n-r)^{\beta-2}\|A^\varepsilon f(r)\|dr \\ &\leq \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha (t-t_n)\|f\|_\varepsilon \int_0^{t_n}(t_n-r)^{\beta(\varepsilon+1)-2}dr + \frac{2}{\mu\pi} C_\alpha(t-t_n)\|f\|_\varepsilon \int_0^{t_n}(t_n-r)^{\beta-2}dr. \end{aligned}$$

Next, we notice that for $\gamma > 0$, we have $\int_0^t (t-r)^{\gamma-1}dr = \Gamma(\gamma)(g_1 * g_\gamma)(t) = \Gamma(\gamma) g_{\gamma+1}(t) = \frac{t^\gamma}{\gamma}$, and therefore,

$$\int_0^{t_n}(t_n-r)^{\beta(\varepsilon+1)-2}dr = \frac{t_n^{\beta(\varepsilon+1)-1}}{\beta(\varepsilon+1)-1} \leq T^\beta \frac{t_n^{\beta\varepsilon-1}}{\beta(\varepsilon+1)-1}, \quad \int_0^{t_n}(t_n-r)^{\beta-2}dr = \frac{t_n^{\beta-1}}{\beta-1} \leq \frac{T^{\beta(1-\varepsilon)}}{\beta-1} t_n^{\beta\varepsilon-1}.$$

Therefore,

$$\|J_1\| \leq \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha (t-t_n)\|f\|_\varepsilon T^\beta \frac{t_n^{\beta\varepsilon-1}}{\beta(\varepsilon+1)-1} + \frac{2}{\mu\pi} C_\alpha (t-t_n)\|f\|_\varepsilon \frac{T^{\beta(1-\varepsilon)}}{\beta-1} t_n^{\beta\varepsilon-1}.$$

By (4.7) we get

$$\int_0^\infty \rho_n^\tau(t)\int_0^{t_n} \|[(g_{\alpha-1} * S_{\alpha,\beta,\mu})(t-r) - (g_{\alpha-1} * S_{\alpha,\beta,\mu})(t_n-r)]f(r)\|drdt \leq C_5 \tau \|f\|_\varepsilon t_n^{\beta\varepsilon-1},$$

where

$$C_5 := \frac{\kappa(M+1)}{\pi}\left(\frac{2}{\mu}\right)^{\varepsilon+1} C_\alpha \frac{T^\beta}{\beta(\varepsilon+1)-1} + \frac{2}{\mu\pi} C_\alpha \frac{T^{\beta(1-\varepsilon)}}{\beta-1}.$$

Now, to estimate J_2 we notice that for $t > 0$ and $x \in D(A^\varepsilon)$ we have as in (4.10) that

$$(g_{\alpha-1} * S_{\alpha,\beta,\mu})(t)x = \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt}}{h(z)} A^{1-\varepsilon}(h(z)-A)^{-1}A^\varepsilon x dz + \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt}}{h(z)} x dz.$$

The inequalities (4.4)-(4.5) and Lemma 4.19 show that

$$\begin{aligned} \|(g_{\alpha-1} * S_{\alpha,\beta,\mu})(t)x\| &\leq \frac{\kappa(M+1)}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|h(z)|^{\varepsilon+1}}\|A^\varepsilon x\||dz| + \frac{1}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|h(z)|}\|x\||dz| \\ &\leq \frac{\kappa(M+1)}{2\pi} C_\alpha \left(\frac{2}{\mu}\right)^{\varepsilon+1} t^{\beta(\varepsilon+1)-1}\|A^\varepsilon x\| + \frac{1}{2\pi}\left(\frac{2}{\mu}\right) t^{\beta-1}\|A^\varepsilon x\|. \end{aligned}$$

Therefore,

$$\int_{t_n}^{t}\|(g_{\alpha-1}*S_{\alpha,\beta,\mu})(t-r)f(r)\|dr \le \frac{\kappa(M+1)}{2\pi}C_\alpha\left(\frac{2}{\mu}\right)^{\varepsilon+1}\|f\|_\varepsilon\int_{t_n}^{t}(t-r)^{\beta(\varepsilon+1)-1}dr + \frac{1}{2\pi}\left(\frac{2}{\mu}\right)\|f\|_\varepsilon\int_{t_n}^{t}(t-r)^{\beta-1}dr.$$

Now, we observe that

$$\int_{t_n}^{t}(t-r)^{\beta(\varepsilon+1)-1}dr = \int_0^t (t-r)^{\beta(\varepsilon+1)-1}dr - \int_0^{t_n}(t-r)^{\beta(\varepsilon+1)-1}dr,$$

and

$$\int_0^t (t-r)^{\beta(\varepsilon+1)-1}dr = \frac{1}{\beta(\varepsilon+1)}t^{\beta(\varepsilon+1)},$$

for all $t\ge 0$. Moreover, the function $x\mapsto x^{\beta(\varepsilon+1)-1}$ is increasing and for $t_n\le t$ we obtain

$$\int_{t_n}^{t}(t-r)^{\beta(\varepsilon+1)-1}dr = \frac{1}{\beta(\varepsilon+1)}t^{\beta(\varepsilon+1)} - \int_0^{t_n}(t-r)^{\beta(\varepsilon+1)-1}dr \le \frac{1}{\beta(\varepsilon+1)}(t^{\beta(\varepsilon+1)}-t_n^{\beta(\varepsilon+1)}).$$

And, analogously

$$\int_{t_n}^{t}(t-r)^{\beta-1}dr \le \frac{1}{\beta}(t^\beta - t_n^\beta).$$

On other hand, by (4.9), $\int_0^\infty \rho_n^\tau(t)(t^{\beta(\varepsilon+1)}-t_n^{\beta(\varepsilon+1)})dt = \frac{\tau^{\beta(\varepsilon+1)}}{n!}\Gamma(n+\beta(\varepsilon+1)+1)-t_n^{\beta(\varepsilon+1)}$, and

$$\begin{aligned} d_n := \frac{\tau^{\beta(\varepsilon+1)}}{n!}\Gamma(n+1+\beta(\varepsilon+1)) &= \tau\tau^{\beta(\varepsilon+1)-1}\frac{\Gamma(n+1+\beta(\varepsilon+1)-1)}{\Gamma(n+2)}(n+1)(n+\beta(\varepsilon+1)) \\ &< t_n t_{n+1} t_{n+1}^{\beta(\varepsilon+1)-2} + \beta(\varepsilon+1)\tau t_{n+1}^{\beta(\varepsilon+1)-1}, \end{aligned}$$

for all $n\in\mathbb{N}$, because $0<\beta(\varepsilon+1)-1<1$ and $\frac{\Gamma(n+1+\eta)}{\Gamma(n+2)}<(n+1)^{\eta-1}$ for all $n\in\mathbb{N}$ and $0<\eta<1$. Moreover, the function $x\mapsto x^{\beta(\varepsilon+1)-2}$ is a decreasing function on $[1,\infty)$, and therefore $t_{n+1}^{\beta(\varepsilon+1)-2}\le t_n^{\beta(\varepsilon+1)-2}$ for all $n\in\mathbb{N}$. This implies that

$$t_{n+1}^{\beta(\varepsilon+1)-1} = (n+1)\tau t_{n+1}^{\beta(\varepsilon+1)-2} \le (n+1)\tau t_n^{\beta(\varepsilon+1)-2} \le t_n^{\beta(\varepsilon+1)-1} + \tau t_n^{\beta(\varepsilon+1)-2} \le 2t_n^{\beta(\varepsilon+1)-1},$$

and $d_n < t_n t_{n+1} t_{n+1}^{\beta(\varepsilon+1)-2} + \beta(\varepsilon+1)\tau t_{n+1}^{\beta(\varepsilon+1)-1} \le t_{n+1}t_n^{\beta(\varepsilon+1)-1} + 2\beta(\varepsilon+1)\tau t_n^{\beta(\varepsilon+1)-1}$, for all $n\in\mathbb{N}$. Since $0<t_n\le T$ and

$$t_{n+1}t_n^{\beta(\varepsilon+1)-1} - t_n^{\beta(\varepsilon+1)} = t_n^{\beta(\varepsilon+1)}\left(\frac{t_{n+1}-t_n}{t_n}\right) = \tau t_n^{\beta(\varepsilon+1)-1},$$

we obtain

$$\int_0^\infty \rho_n^\tau(t)(t^{\beta(\varepsilon+1)}-t_n^{\beta(\varepsilon+1)})dt \quad\le\quad (1+2\beta(\varepsilon+1))\tau T^\beta t_n^{\beta\varepsilon-1}.$$

Similarly, we can prove that

$$\int_0^\infty \rho_n^\tau(t)(t^\beta - t_n^\beta)dt \le (1+2\beta)\tau T^{\beta(1-\varepsilon)}t_n^{\beta\varepsilon-1}.$$

Therefore,

$$\begin{aligned} \int_0^\infty \rho_n^\tau(t)\int_{t_n}^{t}\|(g_{\alpha-1}*S_{\alpha,\beta,\mu})(t-r)f(r)\|drdt &\le \frac{\kappa(M+1)}{2\pi\beta(\varepsilon+1)}C_\alpha\left(\frac{2}{\mu}\right)^{\varepsilon+1}\|f\|_\varepsilon(1+2\beta(\varepsilon+1))\tau T^\beta t_n^{\beta\varepsilon-1} \\ &+ \left(\frac{1}{\mu\pi\beta}\right)\|f\|_\varepsilon(1+2\beta)\tau T^{\beta(1-\varepsilon)}t_n^{\beta\varepsilon-1}, \end{aligned}$$

for all $n \in \mathbb{N}$, and we conclude that

$$\|J_2\| \le C_5' \|f\|_\varepsilon t_n^{\beta\varepsilon-1}.$$

where

$$C_5' := \frac{\kappa(M+1)}{2\pi\beta(\varepsilon+1)} C_\alpha \left(\frac{2}{\mu}\right)^{\varepsilon+1} (1+2\beta(\varepsilon+1))T^\beta + \left(\frac{1}{\mu\pi\beta}\right)(1+2\beta)T^{\beta(1-\varepsilon)}.$$

That is,

$$I_5 \le D_5 \|f\|_\varepsilon t_n^{\beta\varepsilon-1},$$

where $D_5 := C_5 + C_5'$. Summarizing,

$$\|u^n - u(t_n)\| \le (D_1+D_2)\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_0\| + (D_3+D_4)\tau t_n^{\beta\varepsilon-1}\|A^\varepsilon u_1\| + D_5 \tau t_n^{\beta\varepsilon-1}\|f\|_\varepsilon,$$

and we conclude that the constant $C = C(T)$ defined by

$$C := \max\{D_1+D_2, D_3+D_4, D_5\}$$

satisfies

$$\|u^n - u(t_n)\| \le C\tau t_n^{\beta\varepsilon-1}(\|A^\varepsilon u_0\| + \|A^\varepsilon u_1\| + \|f\|_\varepsilon),$$

and the proof is finished. □

5. Some examples

Now, we illustrate the exact solution $u(t)$ at t_n to the fractional differential equation (1.2) and the approximated solution u^n to the difference equation (3.4) given by Theorem 3.17 by applying the families of operators $\{S_{\alpha,\beta,\mu}(t)\}_{t\ge 0}$ and $\{S^n_{\alpha,\beta,\mu}\}_{n\in\mathbb{N}_0}$.

Example 5.22.

Suppose that $A = \rho I$ for some $\rho \in \mathbb{R}$. Then, the Laplace transform of the family $\{S_{\alpha,\beta,\mu}(t)\}_{t\ge 0}$ satisfies

$$\hat{S}_{\alpha,\beta,\mu}(\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \mu\lambda^\beta - \rho},$$

and, by [16, Formula 17.6], we obtain that

$$S_{\alpha,\beta,\mu}(t) = \sum_{j=0}^{\infty} (-\mu)^j t^{(\alpha-\beta)j} E^{j+1}_{\alpha,(\alpha-\beta)j+1}(\rho t^\alpha), \tag{5.1}$$

where, for $p,q,r>0$, $E^r_{p,q}(z)$ is the generalized Mittag-Leffler type function defined by

$$E^r_{p,q}(z) := \sum_{j=0}^{\infty} \frac{(r)_j z^j}{j!\Gamma(pj+q)}, \quad z \in \mathbb{C}.$$

Here, $(r)_j$ denotes the Pochhammer symbol defined by $(r)_j = \frac{\Gamma(r+j)}{\Gamma(r)}$.

Therefore, the solution u to

$$\partial_t^\alpha u(t) + \mu \partial_t^\beta u(t) = \rho u(t) + f(t), \quad t \ge 0, \tag{5.2}$$

with the initial conditions $u(0) = u_0, u_t(0) = u_1$ is given by

(5.3)

$$u(t) = S_{\alpha,\beta,\mu}(t)u_0 + \mu(g_{\alpha-\beta} * S_{\alpha,\beta,\mu})(t)u_0 + (g_1 * S_{\alpha,\beta,\mu})(t)u_1 + \mu(g_{\alpha-\beta+1} * S_{\alpha,\beta,\mu})(t)u_1 + (g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)(t),$$

where $\{S_{\alpha,\beta,\mu}(t)\}_{t\ge 0}$ is defined in (5.1).

On the other hand, by [16, Formula 17.6], it follows that for any $\gamma > 0$,

$$(g_\gamma * S_{\alpha,\beta,\mu})(t) = \sum_{j=0}^{\infty} (-\mu)^j t^{(\alpha-\beta)j+\gamma} E^{j+1}_{\alpha,(\alpha-\beta)j+\gamma+1}(\rho t^\alpha), \quad t \ge 0. \tag{5.4}$$

By Proposition 2.10 we obtain

$$(g_\gamma * S_{\alpha,\beta,\mu})^n = \int_0^\infty \rho_n^\tau(t)(g_\gamma * S_{\alpha,\beta,\mu})(t)dt = \tau \sum_{j=0}^n k_\tau^\gamma(n-j)S^j_{\alpha,\beta,\mu},$$

where

$$S^j_{\alpha,\beta,\mu} = \int_0^\infty \rho_j^\tau(t)S_{\alpha,\beta,\mu}(t)dt.$$

Using (5.1) and [33, Theorem 5.2], we obtain

$$S^j_{\alpha,\beta,\mu} = \sum_{r=0}^\infty \frac{(-\mu)^r}{j!\tau^{j+1}} \int_0^\infty e^{-\frac{1}{\tau}t} t^{(\alpha-\beta)r+j} E^{r+1}_{\alpha,(\alpha-\beta)r+1}(\rho t^\alpha)dt = \sum_{r=0}^\infty \sum_{k=0}^\infty (-\mu)^r \frac{(k+r)!}{k!r!} k_\tau^{(\alpha-\beta)r+\alpha k+1}(j)\rho^k,$$

and, from the semigroup property (2.5), we deduce that

$$(5.5)\qquad \begin{aligned}(k_\tau^\gamma \star S_{\alpha,\beta,\mu})^n &= \sum_{j=0}^n k_\tau^\gamma(n-j)S^j_{\alpha,\beta,\mu} \\ &= \frac{1}{\tau}\sum_{r=0}^\infty \sum_{k=0}^\infty (-\mu)^r \frac{(k+r)!}{k!r!} k_\tau^{(\alpha-\beta)r+\alpha k+\gamma+1}(n)\rho^k, \quad n \in \mathbb{N}.\end{aligned}$$

By Theorem 3.17, the solution u^n to the discrete system

$$(5.6)\qquad {}_C\nabla^\alpha u^n + \mu\, {}_C\nabla^\beta u^n = Au^n + f^n,$$

subject to the initial conditions $u^0 = u_0, u^1 = u_1$, is given by

(5.7)

$$u^n = S^n_{\alpha,\beta,\mu}u_0 + \mu\tau(k_\tau^{\alpha-\beta} \star S_{\alpha,\beta,\mu})^n u_0 + \tau(k_\tau^1 \star S_{\alpha,\beta,\mu})^n u_1 + \mu\tau(k_\tau^{\alpha-\beta+1} \star S_{\alpha,\beta,\mu})^n u_1 + \tau^2(k_\tau^{\alpha-1} \star S_{\alpha,\beta,\mu} \star f)^n,$$

for $n \geq 2$, where for any $\gamma > 0$, $(k_\tau^\gamma \star S_{\alpha,\beta,\mu})^n$ is given in (5.5).

Now, consider the interval $[0, L]$, $L > 0$, and the time step size $\tau = L/N$. As the exact and approximated solutions to (5.2) and (5.6) are expressed in terms of Mittag-Leffler functions (defined as infinite series by (5.3) and (5.7)), the examples consider finite truncations of these series ($M = 80$ terms) for both solutions.

Following [7, Section 5], in our first example, which corresponds to the Bagley-Torvik equation, we set $f(t) = 0$ on $[0, 30]$ and $\alpha = 2, \beta = 3/2, \mu = 1/2, \rho = -1/2$, with initial conditions $u(0) = u_t(0) = 1$. From (5.3) and (5.4) it follows that the solution u is given by

$$\begin{aligned}u(t) = \sum_{j=0}^\infty (-\mu)^j t^{(\alpha-\beta)j} \Big[&E^{j+1}_{\alpha,(\alpha-\beta)j+1}(\rho t^\alpha) + \mu t^{\alpha-\beta} E^{j+1}_{\alpha,(\alpha-\beta)j+\alpha-\beta+1}(\rho t^\alpha) + \\ &t E^{j+1}_{\alpha,(\alpha-\beta)j+2}(\rho t^\alpha) + \mu t^{\alpha-\beta+1} E^{j+1}_{\alpha,(\alpha-\beta)j+\alpha-\beta+2}(\rho t^\alpha)\Big].\end{aligned}$$

In the next example, and following [39, Example 5.2], we take $f(t) = \cos(t)$ and $\alpha = 3/2, \beta = 5/4, \mu = 0.1, \rho = -0.1$. To find an explicit expression to $u(t)$ in (5.3), we just need to determinate $(g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)(t)$. To this end, we begin by expressing $f(t)$ as the series $f(t) = \sum_{q=0}^\infty \frac{(-1)^q}{(2q)!} t^{2q}$. By (5.1), we get

$$(g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)(t) = \sum_{q=0}^\infty \sum_{j=0}^\infty (-1)^q(-\mu)^j \int_0^t \frac{(t-s)^{2q}}{\Gamma(2q+1)} s^{(\alpha-\beta)j+\alpha-1} E^{j+1}_{\alpha,(\alpha-\beta)j+\alpha}(\rho s^\alpha)ds.$$

Using [33, Theorem 2.4] we deduce that

$$(5.8)\qquad \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} s^{p_2-1} E^\gamma_{p_1,p_2}(\rho s^\alpha)ds = t^{\delta+p_2-1} E^\gamma_{p_1,p_2+\delta}(\rho t^\alpha),$$

for any $\alpha, p_1, p_2, \delta > 0$, which implies that

$$(g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)(t) = \sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^q(-\mu)^j \frac{(j+k)!}{j!k!}\frac{t^{(\alpha-\beta)j+\alpha+2q+\alpha k}}{\Gamma((\alpha-\beta)j+\alpha+\alpha k+2q+1)}\rho^k.$$

Now, to determinate $(u^n)_{n=1}^N$ we need to find $(k_\tau^{\alpha-1} \star S_{\alpha,\beta,\mu} \star f)^n$. From Proposition 2.10 we have

$$(k_\tau^{\alpha-1} \star S_{\alpha,\beta,\mu} \star f)^n = \tau \sum_{j=0}^{n} k_\tau^{\alpha-1}(n-j)(S_{\alpha,\beta,\mu} * f)^j = \tau \sum_{j=0}^{n} k_\tau^{\alpha-1}(n-j)\int_0^\infty \rho_j^\tau(t)(S_{\alpha,\beta,\mu} * f)(t)dt,$$

and, by (5.1) and (5.8), we get

$$(S_{\alpha,\beta,\mu} * f)(t) = \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^q(-\mu)^r t^{(\alpha-\beta)r+2q+1}E^{r+1}_{\alpha,(\alpha-\beta)r+2q+2}(\rho t^\alpha).$$

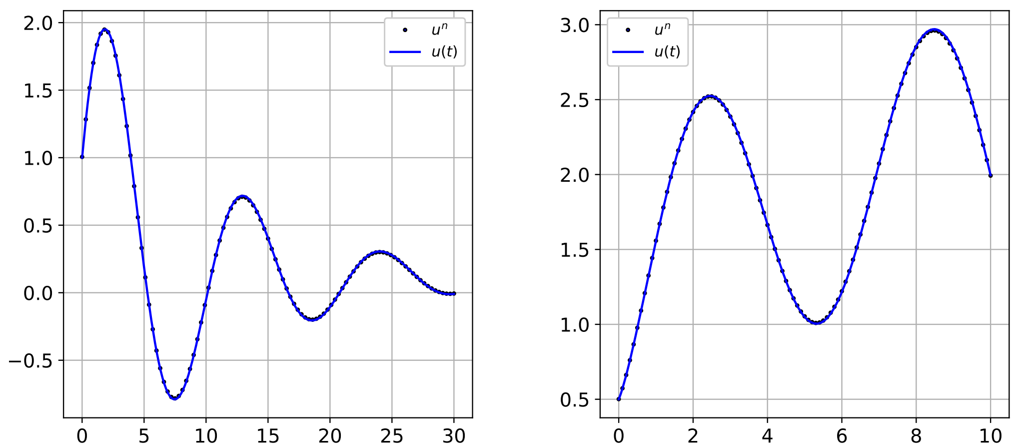
Multiplying this last equation by $\rho_j^\tau(t)$ and integrating over $[0,\infty)$, we use [16, Formula 11.15] to obtain

$$(S_{\alpha,\beta,\mu} * f)^j = \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}(-1)^q(-\mu)^r\frac{(r+k)!}{r!k!}k_\tau^{(\alpha-\beta)r+2q+2+\alpha k}(j)\rho^k.$$

And, by the semigroup property (2.5) and Proposition [23, Proposition 4], we conclude that

$$\begin{aligned}\tau^2(k_\tau^{\alpha-1} \star S_{\alpha,\beta,\mu} \star f)^n &= (g_{\alpha-1} * S_{\alpha,\beta,\mu} * f)^n \\ &= \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}(-1)^q(-\mu)^r\frac{(r+k)!}{r!k!}k_\tau^{(\alpha-\beta)r+2q+1+\alpha k+\alpha}(n)\rho^k.\end{aligned}$$

Figure 3 presents a comparison of the exact solution u and the approximated solution $(u^n)_{n=1}^N$ to the initial value problem defined by (5.2). The exact solution u, given by (5.3), is evaluated at discrete time points $t_n = n\tau$ for $1 \le n \le N$, where $\tau = L/N$ represents the time step. The approximated solution $(u^n)_{n=1}^N$ is obtained using (5.7). This figure illustrates the results for these functions f, for different choices of α, β, L, and, respectively, $N = 120, 100$.



$\alpha = 2$, $\beta = 1.5$, $u(0) = 1$, $u_t(0) = 1$, $f(t) = 0$. $\alpha = 1.5$, $\beta = 1.25$, $u(0) = 1/2$, $u_t(0) = 1/2$, $f(t) = \cos(t)$.

FIGURE 3. Solutions $u(t)$ and u^n for $1 \le n \le N$ on $[0, L]$.

Finally, we compare $u(t_n)$ and u^n to obtain pointwise errors on the interval $[0, L]$. In Figure 4 we show the absolute error for the same functions f and parameters α, β, μ and ρ previously given.

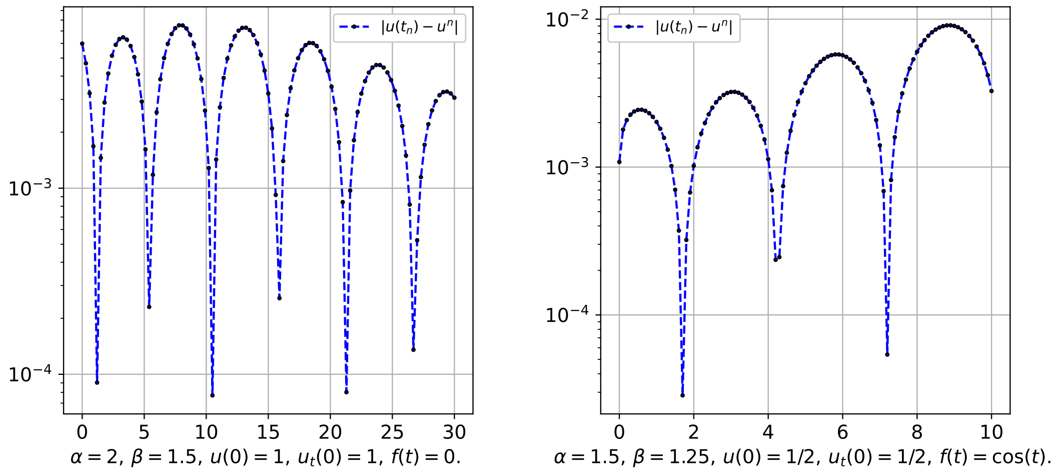


FIGURE 4. Absolute error $|u(t_n) - u^n|$ for $1 \leq n \leq N$.

Here we observe the absolute error estimation, by using the method based on resolvent families and sequences, is consistent with the result given in Theorem 4.21. We observe here a good accuracy using the sequence of operators $\{S^n_{\alpha,\beta,\mu}\}_{n\in\mathbb{N}_0}$ compared with the exact solution given in terms of the resolvent family $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$. Even though the specific examples involve a single variable (scalar case), the method introduced can also be used in more complex situations, such as when dealing with self-adjoint operators (see Example 5.23 below).

Example 5.23.

Now, we consider the following fractional diffusion-wave equation

$$\text{(5.9)} \qquad \begin{cases} \partial_t^2 u(t,x) + \mu \partial_t^{1+\gamma} u(t,x) &= Au(t,x) + f(t,x), \quad x \in \Omega := (-1,1), t > 0, \\ u(0,x) &= u_0(x), \\ u_t(0,x) &= u_1(x), \end{cases}$$

where $u_0, u_1 \in L^2(\Omega)$, $-A$ is a non-negative and self-adjoint operator on the Hilbert space $X = L^2(\Omega)$. If A has a compact resolvent, then $\sigma(A) = \{-\lambda_m : m \in \mathbb{N}\}$, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$ with $\lim_{m\to\infty} \lambda_m = \infty$. If ϕ_m denotes the normalized eigenfunction associated with λ_m, then

$$-Av = \sum_{m=1}^{\infty} \lambda_m \langle v, \phi_m \rangle_{L^2(\Omega)} \phi_m, \text{ for all } v \in D(A).$$

Following [39, Example 5.3], we take the operator $Au(t,x) = \partial_x^2 u(t,x)$, the initial conditions $u_0(x) = u_1(x) = 0$, the function $f(t,x) = e^{-t}\sin(\pi x)$, $\mu = 1$ and $\gamma = 1/2$.

Multiplying both sides of (5.9) by $\phi_m(x)$ and integrating over Ω we get that for every $m \in \mathbb{N}$, the function $u_m(t) := \langle u(t), \phi_m \rangle_{L^2(\Omega)}$ is a solution of

$$\begin{cases} u_m''(t) + \partial_t^{3/2} u_m(t) = -\lambda_m u_m(t) + e^{-t}, \ t > 0 \\ u_m(0) = u_m'(0) = 0. \end{cases}$$

From (5.3) it follows that

$$u_m(t) = (g_{\alpha-1} * S_{\alpha,\beta,\mu} * f_0)(t),$$

where $f_0(t) = e^{-t}$ and

$$S_{\alpha,\beta,\mu}(t) = \sum_{j=0}^{\infty} (-\mu)^j t^{(\alpha-\beta)j} E^{j+1}_{\alpha,(\alpha-\beta)j+1}(-\lambda_m t^\alpha).$$

Since $f_0(t) = \sum_{q=0}^{\infty} \frac{(-t)^q}{q!}$ we may proceed similarly to Example 5.22 to obtain

$$u_m(t) = (g_{\alpha-1} * S_{\alpha,\beta,\mu} * f_0)(t) = \sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-1)^q(-\mu)^j \frac{(j+k)!}{j!k!} \frac{t^{(\alpha-\beta)j+\alpha+q+\alpha k}}{\Gamma((\alpha-\beta)j+\alpha+\alpha k+q+1)}(-\lambda_m)^k.$$

Since

$$u(t,x) = \sum_{m=1}^{\infty} u_m(t)\phi_m(x),\ \forall\, t \geq 0,\ x \in \Omega,$$

we get that the explicit analytical solutions to (5.9) is given by

$$u(t,x) = \sum_{m=1}^{\infty} (g_{\alpha-1} * S_{\alpha,\beta,\mu} * f_0)(t)\phi_m(x),\ \ \forall\, t \geq 0,\ x \in \Omega,$$

where $\alpha = 2, \beta = 3/2$ and $\mu = 1$. Finally, and proceeding as in Example 5.22, we may obtain that

$$\tau^2(k_\tau^{\alpha-1} \star S_{\alpha,\beta,\mu} \star f_0)^n = \sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-1)^q(-\mu)^j \frac{(j+k)!}{j!k!} k_\tau^{(\alpha-\beta)j+\alpha+q+1+\alpha k}(n)\rho^k(-\lambda_m)^k.$$

and therefore, the solution to the semi-discrete problem

$$\nabla^2 u^n(x) +_C \nabla^{3/2}u^n(x) = Au^n(x) + f^n(x), \quad x \in \Omega := (-1,1), n \in \mathbb{N}_0,$$

with initial conditions $u^0(x) = u^1(x) = 0, x \in \Omega$, is given by

$$u^n(x) = \sum_{m=1}^{\infty}\sum_{q=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-1)^q(-\mu)^j \frac{(j+k)!}{j!k!} k_\tau^{(\alpha-\beta)j+\alpha+q+1+\alpha k}(n)\rho^k(-\lambda_m)^k\phi_m(x), \quad x \in (-1,1).$$

6. Appendix

A. **Resolvent families.** This section provides a summary of the main properties of the resolvent families and sequences employed throughout this paper.

The following result, similar to the Hille-Yosida Theorem for C_0-semigroups, follows directly from [20, Theorem 3.4].

Theorem A.1. *Let A be a closed linear densely defined operator in a Banach space X. Let $\mu \geq 0$ and $1 < \beta < \alpha \leq 2$. Then, the following assertions are equivalent.*

(1) *The operator A is the generator of an (α,β,μ)-resolvent family $\{S_{\alpha,\beta,\mu}(t)\}_{t\geq 0}$ which satisfies $\|S_{\alpha,\beta,\mu}(t)\| \leq Ke^{\nu t}$ for all $t \geq 0$ and for some constants $K > 0$ and $\nu \in \mathbb{R}$.*

(2) *There exist constants $\nu \in \mathbb{R}$ and $K > 0$ such that*
   (a) *$\{\lambda^\alpha + \mu\lambda^\beta : \mathrm{Re}\lambda > \nu\} \subset \rho(A)$ and*
   (b) *The mapping $\lambda \mapsto H(\lambda) := \lambda^{\alpha-1}\left(\lambda^\alpha + \mu\lambda^\beta - A\right)^{-1}$ satisfies the estimates*

$$\|H^{(n)}(\lambda)\| \leq \frac{Kn!}{(\lambda-\nu)^{n+1}},$$

for all $\lambda > \nu$ and $n = 0, 1, 2...$, where $H^{(n)}(\lambda) = \frac{d^n H(\lambda)}{d\lambda^n}$.

Proposition A.2 (Generation)**.** *Let $\mu \geq 0$ and $1 < \beta < \alpha \leq 2$ and $A \in \mathrm{Sect}(\theta, M)$ where $\theta = \frac{\alpha\pi}{2}$. If $(\alpha-\beta) \leq 1$, then A generates an (α,β,μ)-resolvent family.*

Proof. By Theorem A.1 we need to find constants $K > 0$ and $\nu \in \mathbb{R}$ satisfying condition (2). If fact, for $\lambda \in \mathbb{C}$ we define $h(\lambda) := \lambda^{\alpha} + \mu\lambda^{\beta}$. Let $\lambda = re^{i\phi}$ with $|\phi| < \frac{\pi}{2}$ and $r > 0$. We may assume that $\phi \geq 0$ without any restriction. Then

$$\arg(h(re^{i\phi})) = \mathrm{Im}(\ln(h(re^{i\phi}))) = \mathrm{Im}\int_0^{\phi} \frac{d}{dt}\ln(h(re^{it}))dt = \mathrm{Im}\int_0^{\phi} \frac{h'(re^{it})ire^{it}}{h(re^{it})}dt.$$

Since $\frac{\lambda h'(\lambda)}{h(\lambda)} = (\alpha-\beta)\frac{\lambda^{\alpha}}{\lambda^{\alpha}+\mu\lambda^{\beta}} + \beta$, and $\cos(\phi(\alpha-\beta)) > 0$ we obtain $\frac{|r^{\alpha-\beta}e^{i\phi(\alpha-\beta)}|}{|r^{\alpha-\beta}e^{i\phi(\alpha-\beta)}+\mu|} \leq 1$ for all $r > 0$ and therefore

$$|\arg(h(\lambda))| \leq \int_0^{\phi}\left((\alpha-\beta)\frac{|r^{\alpha-\beta}e^{i\phi(\alpha-\beta)}|}{|r^{\alpha-\beta}e^{i\phi(\alpha-\beta)}+\mu|} + \beta\right)dt \leq \alpha\phi < \frac{\alpha\pi}{2} = \theta.$$

As A is sectorial operator, $h(\lambda) \in S_{\theta}$ for all $\lambda > \nu := 0$, and therefore $h(\lambda) \in \rho(A)$. For such λ we define $H(\lambda) := \lambda^{\alpha-1}(h(\lambda) - A)^{-1}$. Then, $H(\lambda) = \frac{\lambda^{\alpha-1-\beta}}{\lambda^{\alpha-\beta}+\mu}h(\lambda)(h(\lambda)-A)^{-1}$. Since $(\alpha-\beta) \leq 1$, $A \in \mathrm{Sect}(\theta, M)$ and $g(\lambda) \in \rho(A)$, we obtain

$$\|\lambda H(\lambda)\| \leq \frac{|\lambda^{\alpha-\beta}|}{|\lambda^{\alpha-\beta}+\mu|}|h(\lambda)|\,\|(h(\lambda)-A)^{-1}\| \leq M. \tag{A.1}$$

On the other hand, $\lambda^2 H'(\lambda) = (\alpha-1)\lambda H(\lambda) - \alpha(\lambda H(\lambda))^2 - \beta\mu\lambda^{\beta-\alpha}\lambda H(\lambda)\lambda H(\lambda)$. From (A.1) we obtain that $\|\lambda^{\beta-\alpha}\lambda H(\lambda)\| \leq \frac{M}{|\lambda^{\alpha-\beta}+\mu|}$, which implies

$$\|\lambda^2 H'(\lambda)\| \leq (\alpha-1)M + \alpha M^2 + \beta\mu\frac{M}{|\lambda^{\alpha-\beta}+\mu|}M \leq (\alpha-1)M + \alpha M^2 + \beta M^2 =: K, \tag{A.2}$$

for all $\lambda > 0$. From (A.1)-(A.2) we conclude that A is the generator of an (α,β,μ)-resolvent family such that $\|S_{\alpha,\beta,\mu}(t)\| \leq K$, by Proposition [30, Proposition 0.1] and Theorem A.1. □

The next theorem gives an asymptotic behavior of $\|S_{\alpha,\beta,\mu}(t)\|$. Its proof follows similarly to [17, Theorem 4.1] and therefore, we omit the details.

Theorem A.3. *Let $\mu \geq 0$ and $1 < \beta < \alpha \leq 2$ and $A \in \mathrm{Sect}(\theta, \omega, M)$ where $\theta = \frac{\alpha\pi}{2}$ and $\omega < 0$. If $(\alpha-\beta) \leq \frac{1}{2}$, then there exists a constant $C > 0$ depending only on α, β and μ such that*

$$\|S_{\alpha,\beta,\mu}(t)\| \leq \frac{C}{1+|\omega|(t^{\alpha}+\mu t^{\beta})}, \quad t \geq 0. \tag{A.3}$$

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