DETERMINING THE ORDER IN ABSTRACT FRACTIONAL SYSTEMS

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ABSTRACT. In this paper we identify the order $\alpha > 0$ in the abstract fractional differential equation $\partial^\alpha u(t) = Au(t)$, where the time-fractional derivative ∂^α is understood in the sense of Caputo and Riemann-Liouville, A is a closed (possibly unbounded) linear operator in a Banach space X, and $0 < \alpha < 1$ or $1 < \alpha < 2$.

To this end, we propose a method based on the asymptotic behavior of specific families of linear operators generated by A, enabling the determination of the parameter α beyond the classical self-adjoint setting. We also provide examples to illustrate the resulting reconstruction formulas.

1. Introduction

The problem of finding or approximating the order in time-fractional differential equations has been widely studied over the last decade. See for instance the monographs [19, 20].

For $0 < \alpha < 1$ consider the fractional differential equation for the Caputo fractional derivative

$$\partial_t^\alpha u(x,t) = Au(x,t), \quad x \in \Omega, t > 0, \tag{1.1}$$

under the initial condition $u(x,0) = u_0(x), x \in \Omega$, where Ω and A are defined as follows: For a bounded open set $\Omega \subset \mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, let X be the Hilbert space $L^2(\Omega)$. On X, the operator $\mathcal{A}$ is defined by $\mathcal{A}u(x) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N A_{ij}(x)\frac{\partial}{\partial x_i}u(x)\right)$, $u \in X$, where $A_{ij} = A_{ji}$ for any $1 \le i,j \le N$. Suppose that there exists a constant $\gamma > 0$ such that $\sum_{i,j=1}^N A_{ij}(x)\xi_i\xi_j \ge \gamma|\xi|^2$, for all $\xi \in \mathbb{R}^N$ and $x \in \overline{\Omega}$. The operator $A : D(A) \to X$ is defined by

$$(Au)(x) = (\mathcal{A}u)(x), \quad x \in \Omega,$$

where $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. The operator $-A$ has a discrete spectrum and its eigenvalues satisfy $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ and $\lim_{n\to\infty}\lambda_n = \infty$. Now, let $\phi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ be the normalized eigenfunction associated with $-\lambda_n$. Then, by the Fourier method, the solution u to (1.1) is given by

$$u(x,t) = \sum_{n=1}^\infty \langle u_0, \phi_n\rangle_{L^2(\Omega)} E_{\alpha,1}(-\lambda_n t^\alpha)\phi_n(x), \quad x \in \Omega, t > 0, \tag{1.2}$$

where for any $\alpha, \beta > 0$ and $z \in \mathbb{C}$, $E_{\alpha,\beta}$ denotes the Mittag-Leffler function which, is defined by $E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}$. If $u(t,x)$ denotes the solution to (1.1), $u_0 \in C_0^\infty(\Omega)$ with $Au_0(x) \neq 0$, and x_0 is a fixed element in Ω, then the order α in (1.1) is given by (see [16, Theorem 1])

$$\alpha = \lim_{t\to 0^+} \frac{t\frac{\partial u}{\partial t}(x_0,t)}{u(x_0,t) - u_0(x_0)}. \tag{1.3}$$

See also [4, 6, 26, 31, 33, 38] for some related results.

Now, if for any $t \ge 0$, we define the family of linear operators $S_{\alpha,\beta}(t) : X \to X$ by

$$S_{\alpha,\beta}(t)u(x) := \sum_{n=1}^\infty \langle u_0, \phi_n\rangle_{L^2(\Omega)} t^{\beta-1}E_{\alpha,\beta}(-\lambda_n t^\alpha)\phi_n(x), \quad x \in \Omega, u \in X,$$

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then the solution (1.2) to equation (1.1) can be written as

$$u(x,t) = S_{\alpha,1}(t)u_0(x), \quad x \in \Omega, t \geq 0. \tag{1.4}$$

Thus, the order α given in (1.3) can be written using the family of linear operators $\{S_{\alpha,1}(t)\}_{t\geq 0}$ as

$$\alpha = \lim_{t\to 0^+} \frac{tS'_{\alpha,1}(t)u_0(x_0)}{S_{\alpha,1}(t)u_0(x_0) - u_0(x_0)}, \quad x_0 \in \Omega.$$

The properties of the Laplace transform of the Mittag-Leffler function imply that the family of operators $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ corresponds to an (α,β)-fractional resolvent family generated by A, see for instance [23, 28]. This theory allows us to write the solutions to fractional differential equations (for the Caputo and Riemann-Liouville derivatives) in case $0 < \alpha < 1$ and $1 < \alpha < 2$ as a variation of parameters formula. In fact, consider the fractional differential equations for the Caputo fractional derivative,

$$\begin{cases} \partial_t^\alpha u(x,t) &= Au(x,t), \quad t > 0 \\ u(x,t) &= 0, \quad x \in \partial\Omega, t > 0 \\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{cases} \tag{1.5}$$

(for $0 < \alpha < 1$) and

$$\begin{cases} \partial_t^\alpha u(x,t) &= Au(x,t), \quad t > 0 \\ u(x,t) &= 0, \quad x \in \partial\Omega, t > 0 \\ u(x,0) &= u_0(x), \quad x \in \Omega \\ \partial_t u(x,0) &= u_1(x), \quad x \in \Omega, \end{cases} \tag{1.6}$$

(for $1 < \alpha < 2$), where $u_0, u_1 \in X$. By [18], the solution to (1.5) is given by (1.4) and the solution to (1.6) is

$$u(x,t) = \sum_{n=1}^\infty \langle u_0, \phi_n\rangle_{L^2(\Omega)} E_{\alpha,1}(-\lambda_n t^\alpha)\phi_n(x) + \sum_{n=1}^\infty \langle u_1, \phi_n\rangle_{L^2(\Omega)} tE_{\alpha,2}(-\lambda_n t^\alpha)\phi_n(x),$$

and therefore, the solutions to (1.5) and (1.6) can be written, in terms of the resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$, respectively, as

$$u(x,t) = S_{\alpha,1}(t)u_0(x), \quad u(x,t) = S_{\alpha,1}(t)u_0(x) + S_{\alpha,2}(t)u_1(x), \quad x \in \Omega, t \geq 0. \tag{1.7}$$

Similarly, if we consider the fractional differential equations for the Riemann-Liouville fractional derivative

$$\begin{cases} {}^R\partial_t^\alpha u(x,t) &= Au(x,t), \quad t > 0 \\ u(x,t) &= 0, \quad x \in \partial\Omega, t > 0 \\ (g_{1-\alpha} * u)(x,0) &= u_0(x), \quad x \in \Omega, \end{cases} \tag{1.8}$$

(for $0 < \alpha < 1$) and

$$\begin{cases} {}^R\partial_t^\alpha u(x,t) &= Au(x,t), \quad t > 0 \\ u(x,t) &= 0, \quad x \in \partial\Omega, t > 0 \\ (g_{2-\alpha} * u)(x,0) &= u_0(x), \quad x \in \Omega \\ \partial_t(g_{2-\alpha} * u)(x,0) &= u_1(x), \quad x \in \Omega, \end{cases} \tag{1.9}$$

(for $1 < \alpha < 2$), where $u_0, u_1 \in X$. Here ${}^R\partial_t^\alpha$ corresponds to the Riemann-Liouville fractional derivative and $g_{2-\alpha}(t) := t^{1-\alpha}/\Gamma(2-\alpha)$, then the solutions to (1.8) and (1.9) are given respectively by (see [13, 34, 37]),

$$u(x,t) = \sum_{n=1}^\infty \langle u_0, \phi_n\rangle_{L^2(\Omega)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\phi_n(x), \text{ and}$$

$$u(x,t) = \sum_{n=1}^\infty \langle u_0, \phi_n\rangle_{L^2(\Omega)} t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)\phi_n(x) + \sum_{n=1}^\infty \langle u_1, \phi_n\rangle_{L^2(\Omega)} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\phi_n(x),$$

which can be written, respectively, as

$$u(x,t) = S_{\alpha,\alpha}(t)u_0(x), \quad u(x,t) = S_{\alpha,\alpha-1}(t)u_0(x) + S_{\alpha,\alpha}(t)u_1(x), \quad x \in \Omega. \tag{1.10}$$

Resolvent families have been extensively studied, both in abstract settings and in applications (see for instance [9, 10, 11, 14, 15, 17, 18, 28, 30, 29, 34, 36]). The operators $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ are well-known in some cases: the uniqueness of the Laplace transform implies that $\{S_{1,1}(t)\}_{t\geq 0}$ is the C_0-semigroup generated by A, $\{S_{2,1}(t)\}_{t\geq 0}$ and $\{S_{2,2}(t)\}_{t\geq 0}$ are, respectively, the cosine and sine family generated by A, see [5]. Now, for $1\leq\alpha\leq 2$ and $\beta=1$, $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is an α-times resolvent [24], and the case $1\leq\alpha=\beta\leq 2$ corresponds to an α-order resolvent (see [25]).

In this paper, we propose a method based on the exploration of the resolvent families $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ to identify the order α in the fractional differential equations (1.5)–(1.6) and (1.8)–(1.9), where A is a closed linear operator in a Banach space X. More specifically, we consider the abstract fractional differential equation for the Caputo fractional derivative

$$(1.11)\qquad \begin{cases} \partial_t^\alpha v(t) &= Av(t), \quad t>0,\\ v(0) &= x,\end{cases}$$

where $0<\alpha<1$, A is a closed linear operator defined in a Banach space X and $x\in X$. If $v(t;x)=S_{\alpha,1}(t)x$ is the solution to Problem (1.11) (where $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is the $(\alpha,1)$-resolvent family generated by A), then we prove in Theorem 3.4 that: if $x\in D(A)$, and $x^*\in X^*$ with $x^*(Ax)\neq 0$, then

$$\alpha=\lim_{t\to 0^+}\frac{x^*(t\psi_t(x))}{x^*(\varphi_t(x))},$$

where the functions $\varphi_t: X\to X$ and $\psi_t: X\to X$, are defined respectively, by $\varphi_t(x)=v(t;x)-x$ and $\psi_t(x)=v'(t;x)$. We notice that this formula corresponds exactly to the abstract version of (1.3). Notably, the abstract framework proposed in this article allows for the determination of the parameter α in more general settings beyond the self-adjoint case in Hilbert spaces. This represents a significant extension of results previously established in the literature

Moreover, we obtain here similar results for (1.11) in case $1<\alpha<2$ and for the abstract fractional differential equation for the Riemann-Liouville fractional derivative in case $0<\alpha<1$ and $1<\alpha<2$, which correspond to completely new results in the literature.

The paper is organized as follows. In Section 2 we give the preliminaries on fractional calculus and fractional resolvent families generated by a closed linear operator A. In Section 3 we identify $\alpha\in(0,1)$ for small times for the Caputo and Riemann-Liouville fractional derivatives. Section 4 is devoted to the same problem, but with $\alpha\in(1,2)$. Finally, we illustrate our results with some examples.

2. Preliminaries

Let $X\equiv(X,\|\cdot\|)$ be a Banach space. By $\mathcal{B}(X)$ we denote the Banach space of all bounded and linear operators from X into X. For a given closed linear operator A on X, $\rho(A)$ denotes its resolvent set and $R(\lambda,A)=(\lambda-A)^{-1}$ its resolvent operator, which is defined for all $\lambda\in\rho(A)$.

A strongly continuous family of linear operators $\{S(t)\}_{t\geq 0}\subset\mathcal{B}(X)$ is called *exponentially bounded* if there exist constants $M>0$ and $w\in\mathbb{R}$ such that $\|S(t)\|\leq Me^{wt}$, for all $t>0$.

For a given $\alpha>0$, we define the function g_α as $g_\alpha(t):=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\Gamma(\cdot)$ stands for the Gamma function. It is easy to see that if $\alpha,\beta>0$, then the functions g_γ satisfy the semigroup law $g_{\alpha+\beta}(t)=(g_\alpha*g_\beta)(t)$, where $(f*g)$ denotes the finite convolution $(f*g)(t):=\int_0^t f(t-s)g(s)ds$.

For $n-1<\alpha<n$, where $n\in\mathbb{N}$, the Caputo and Riemann-Liouville fractional derivatives of order α of a function f are defined, respectively, by

$$\partial_t^\alpha f(t):=(g_{n-\alpha}*f')(t)=\int_0^t g_{n-\alpha}(t-s)f^{(n)}(s)ds,\quad {}^R\partial_t^\alpha f(t):=\frac{d^n}{dt^n}\int_0^t g_{n-\alpha}(t-s)f(s)ds.$$

If $\alpha=1$ or $\alpha=2$, then $\partial_t^1={}^R\partial_t^1=\frac{d}{dt}$ and $\partial_t^2={}^R\partial_t^2=\frac{d^2}{dt^2}$. For more details, examples and applications on fractional calculus, we refer to the reader to [22]. In this paper, our focus is in $0<\alpha<1$ and $1<\alpha<2$.

Definition 2.1. *Let A be a closed and linear operator defined on a Banach space X. Given $\alpha,\beta>0$ we say that A is the generator of an (α,β)-resolvent family, if there exist $\omega\geq 0$ and a strongly continuous* *function* $S_{\alpha,\beta}:(0,\infty)\to\mathcal{B}(X)$ *such that* $S_{\alpha,\beta}(t)$ *is exponentially bounded,* $\{\lambda^\alpha : \mathrm{Re}\lambda>\omega\}\subset\rho(A)$, *and for all* $x\in X$,

$$\lambda^{\alpha-\beta}\left(\lambda^\alpha - A\right)^{-1}x = \int_0^\infty e^{-\lambda t}S_{\alpha,\beta}(t)x\,dt, \;\; \mathrm{Re}\lambda>\omega. \tag{2.12}$$

In this case, $\{S_{\alpha,\beta}(t)\}_{t>0}$ *is called the* (α,β)-resolvent family *generated by* A.

Moreover, if an operator A with domain $D(A)$ is the infinitesimal generator of $S_{\alpha,\beta}(t)$, then for $x\in D(A)$ we have

$$Ax = \lim_{t\to 0^+}\frac{S_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)}.$$

We notice that the case $S_{1,1}(t)$ corresponds to a C_0-semigroup, $S_{2,1}(t)$ is a cosine family and $S_{2,2}(t)$ is a sine family generated by A. In the scalar case, that is, when $A=\rho I$, where $\rho\in\mathbb{C}$ and I denotes the identity operator, we have, by the uniqueness of the Laplace transform, that $S_{\alpha,\beta}(t)$ corresponds to the function $t^{\beta-1}E_{\alpha,\beta}(\rho t^\alpha)$.

Now, we give some examples of (α,β)-resolvent families. For $0<\alpha<1$ and $\beta\ge\alpha$, let $\{S_{\alpha,\beta}(t)\}_{t\ge 0}$ be the family of operators defined by

$$S_{\alpha,\beta}(t)f(s) := \int_0^s f(s-r)\varphi_{\alpha,\beta-\alpha}(t,r)\,dr,$$

where $s\in\mathbb{R}_+$, $f\in L^1(\mathbb{R}_+)$ and the function $\varphi_{a,b}(t,r)$ is defined by

$$\varphi_{a,b}(t,r) := t^{b-1}W_{-a,b}(-rt^{-a}), \quad a>0, b\ge 0,$$

where $W_{-a,b}(z) := \sum_{n=0}^\infty \frac{z^n}{n!\Gamma(-an+b)}$ $(z\in\mathbb{C})$ denotes the Wright function. Then, $\{S_{\alpha,\beta}(t)\}_{t\ge 0}$ is an (α,β)-resolvent family on the Banach space $X=L^1(\mathbb{R}_+)$ generated by $A=-\frac{d}{dt}$. See [3, Example 11].

On the other hand, if $0<\alpha<1$, and A is a generator of a C_0-semigroup $\{T(t)\}_{t\ge 0}$, then by a subordination principle ([2, 7]), A is the generator of the $(\alpha,1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\ge 0}$ given by

$$S_{\alpha,1}(t)x = \int_0^\infty \Phi_\alpha(r)T(rt^\alpha)x\,dr, \quad t\ge 0, x\in X,$$

where Φ_α is the Wright type function ([32, Appendix F]) $\Phi_\alpha(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\alpha n+1-\alpha)} = \int_\gamma \mu^{\alpha-1}e^{\mu-z\mu^\alpha}d\mu$, where γ is a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise. In addition, by [21, Theorem 3.1] A is also the generator of an (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\ge 0}$ given by

$$S_{\alpha,\alpha}(t)x = \alpha\int_0^\infty t^{\alpha-1}r\Phi_\alpha(r)T(rt^\alpha)x\,dr, \quad t\ge 0, x\in X.$$

Now, let $1<\alpha<2$. Assume that A generates a cosine family $\{C(t)\}$ defined in X. Then A is the generator of the $(\alpha,1)$ resolvent family $\{S_{\alpha,1}(t)\}_{t\ge 0}$ defined by

$$S_{\alpha,1}(t)x := \int_0^\infty \psi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s)C(s)x\,ds, \quad t\ge 0, x\in X, \tag{2.13}$$

where $\psi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}$ is the Wright type function given by

$$\begin{aligned}\psi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s) \;&=\; \frac{1}{\pi}\int_0^\infty \rho^{\frac{\alpha}{2}-1}e^{-s\rho^{\frac{\alpha}{2}}\cos\frac{\alpha}{2}(\pi-\theta)-t\rho\cos\theta}\times\\ &\qquad\qquad \sin\left(t\rho\sin\theta - s\rho^{\frac{\alpha}{2}}\sin\tfrac{\alpha}{2}(\pi-\theta)+\tfrac{\alpha}{2}(\pi-\theta)\right)d\rho,\end{aligned}$$

for $\theta\in(\pi-\frac{2}{\alpha},\pi/2)$. In addition, it also generates an (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}$ defined by

$$S_{\alpha,\alpha}(t)x := \int_0^\infty \psi_{\frac{\alpha}{2},\frac{\alpha}{2}}(t,s)C(s)x\,ds, \quad t\ge 0, x\in X,$$

where $\psi_{\frac{\alpha}{2},\frac{\alpha}{2}}$ is the Wright type function given by

$$\psi_{\frac{\alpha}{2},\frac{\alpha}{2}}(t,s) = (g_{\frac{\alpha}{2}} * \psi_{\frac{\alpha}{2},0}(\cdot,s))(t),$$

where $\psi_{\frac{\alpha}{2},0}(\cdot,s)$ is given by

$$\psi_{\alpha,0}(t,s)=\frac{1}{\pi}\int_0^\infty e^{t\rho\cos\theta-s\rho^\alpha\cos\alpha\theta}\cdot\sin(t\rho\sin\theta-s\rho\sin\alpha\theta+\theta)d\rho,$$

for $\pi/2<\theta<\pi$. An example of such operators includes $A=\Delta$ on $L^2(\Omega)$ with domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, where Ω is an open bounded set in $\mathbb{R}^N$, see for instance [35, Section 5].

From [1, 3] or [27] we have the following result that gives some important properties of $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$.

Proposition 2.2. *If $\alpha,\beta>0$ and A generates an (α,β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t>0}$, then*

(1) $\lim_{t\to 0^+}\dfrac{S_{\alpha,\beta}(t)x}{g_\beta(t)}=x$, *for all* $x\in X$.
(2) $S_{\alpha,\beta}(t)x\in D(A)$ *and* $S_{\alpha,\beta}(t)Ax=AS_{\alpha,\beta}(t)x$ *for all* $x\in D(A)$ *and* $t>0$.
(3) *For all* $x\in D(A)$, $S_{\alpha,\beta}(t)x=g_\beta(t)x+\int_0^t g_\alpha(t-s)AS_{\alpha,\beta}(s)xds$.
(4) $\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds\in D(A)$ *and*

$$S_{\alpha,\beta}(t)x=g_\beta(t)x+A\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds,$$

for all $x\in X$.

For a given locally integrable function $f:[0,\infty)\to X$, we define the *Laplace transform* of f, denoted by $\hat{f}(\lambda)$ (or $\mathcal{L}(f)(\lambda)$) as $\hat{f}(\lambda)=\int_0^\infty e^{-\lambda t}f(t)dt$, provided the integral converges for some $\lambda\in\mathbb{C}$.

The following lemma will be useful for our purposes.

Lemma 2.3. *Assume that A is the generator of the family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$. If $\gamma>0$, then A generates the family $\{S_{\alpha,\beta+\gamma}(t)\}_{t\geq 0}$ given by*

$$S_{\alpha,\beta+\gamma}(t)=(g_\gamma * S_{\alpha,\beta})(t). \tag{2.14}$$

Proof. In fact, for any $\gamma>0$ and $\lambda^\alpha\in\rho(A)$ with $\mathrm{Re}\lambda>w$, we have

$$\mathcal{L}(g_\gamma * S_{\alpha,\beta})(\lambda)=\lambda^{\alpha-\beta-\gamma}(\lambda^\alpha-A)^{-1}=\lambda^{\alpha-(\beta+\gamma)}(\lambda^\alpha-A)^{-1}=\hat{S}_{\alpha,\beta+\gamma}(\lambda).$$

And the result follows from the uniqueness of the Laplace transform. □

3. Determination of α for small times. The subdiffusion case: $0<\alpha<1$.

Let $0<\alpha<1$. In this section we determine the order α of the fractional differential equations $\partial_t^\alpha u(t)=Au(t)$ and ${}^R\partial_t^\alpha u(t)=Au(t)$, for the Caputo and Riemann-Liouville fractional derivatives. Here A is a given closed linear operator.

Let $x\in X$. Consider the equation for the Caputo fractional derivative

$$\begin{cases}\partial_t^\alpha u(t) &= Au(t), \quad t>0\\ u(0) &= x\end{cases} \tag{3.15}$$

where $0<\alpha<1$. For each $x\in X$, we denote by $u(t;x)$ the solution to Problem (3.15). If $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is the $(\alpha,1)$-resolvent family generated by A, then,

$$u(t;x)=S_{\alpha,1}(t)x,$$

see for instance [7, Chapter 1]. Next, we define the operators $\varphi_t:X\to X$ and $\psi_t:X\to X$, by

$$\varphi_t(x)=u(t;x)-x \quad\text{and}\quad \psi_t(x)=u'(t;x), \tag{3.16}$$

where $u(t;x)$ is the solution of (3.15).

We consider the following inverse problem:

- Let $x\in X$ be fixed. Determine $\alpha\in(0,1)$ in (3.15) from the observation data $u(t;x)$ for small t.

The next result gives an answer for the Caputo fractional derivative.

Theorem 3.4. *Suppose that A generates the $(\alpha,1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$. Let $x\in D(A)$. If $x^*\in X^*$ with $x^*(Ax)\neq 0$, then*

$$\alpha=\lim_{t\to 0^+}\frac{x^*(t\psi_t(x))}{x^*(\varphi_t(x))},$$

where ψ_t and φ_t are defined in (3.16).

Proof. Let $x\in D(A)$ and $x^*\in X^*$ with $x^*(Ax)\neq 0$. Since $u(t;x)=S_{\alpha,1}(t)x$, from Proposition 2.2 and Lemma 2.3 we can write

$$u(t;x)-x=A(g_\alpha * S_{\alpha,1})(t)x=AS_{\alpha,\alpha+1}(t)x, \tag{3.17}$$

for any $x\in X$. Therefore

$$t^{-\alpha}\varphi_t(x)=t^{-\alpha}(u(t;x)-x)=\frac{AS_{\alpha,\alpha+1}(t)}{g_{\alpha+1}(t)}\frac{1}{\Gamma(\alpha+1)}x. \tag{3.18}$$

By Proposition 2.2 we have for any $x\in D(A)$ that

$$\lim_{t\to 0^+}x^*(t^{-\alpha}\varphi_t(x))=\frac{1}{\Gamma(\alpha+1)}x^*(Ax). \tag{3.19}$$

On the other hand, for any $\lambda^\alpha\in\rho(A)$, we have by Proposition 2.2 that

$$\widehat{S}'_{\alpha,1}(\lambda)x=\lambda\hat{S}_{\alpha,1}(\lambda)x-S_{\alpha,1}(0)x=\lambda^\alpha(\lambda^\alpha-A)^{-1}x-x=A(\lambda^\alpha-A)^{-1}x=A\hat{S}_{\alpha,\alpha}(\lambda)x. \tag{3.20}$$

By Proposition 2.2 and Lemma 2.3 we have for any $x\in D(A)$ that

$$u'(t;x)=S'_{\alpha,1}(t)x=AS_{\alpha,\alpha}(t)x=g_\alpha(t)Ax+A(g_\alpha * S_{\alpha,\alpha})(t)Ax=g_\alpha(t)Ax+AS_{\alpha,2\alpha}(t)Ax. \tag{3.21}$$

Therefore,

$$t^{1-\alpha}\psi_t(x)=t^{1-\alpha}u'(t;x)=\frac{Ax}{\Gamma(\alpha)}+At^{1-\alpha}S_{\alpha,2\alpha}(t)Ax.$$

Proposition 2.2 implies again that

$$\lim_{t\to 0^+}t^{1-\alpha}AS_{\alpha,2\alpha}(t)x=\lim_{t\to 0^+}\frac{S_{\alpha,2\alpha}(t)}{g_{2\alpha}(t)}\frac{t^\alpha}{\Gamma(2\alpha)}Ax=0,$$

where the limit is understood in X. Hence, we obtain

$$\lim_{t\to 0^+}x^*(t^{1-\alpha}u'(t;x))=\lim_{t\to 0^+}x^*(t^{1-\alpha}\psi_t(x))=\frac{x^*(Ax)}{\Gamma(\alpha)}, \tag{3.22}$$

for $x\in D(A)$. By (3.19) and (3.22) we obtain

$$\lim_{t\to 0^+}\frac{x^*(t\psi_t(x))}{x^*(\varphi_t(x))}=\lim_{t\to 0^+}\frac{x^*(t^{1-\alpha}\psi_t(x))}{x^*(t^{-\alpha}\varphi_t(x))}=\frac{\frac{x^*(Ax)}{\Gamma(\alpha)}}{\frac{x^*(Ax)}{\Gamma(\alpha+1)}}=\alpha.$$

□

Remark 3.5. *Theorem 3.4 says, in the scalar-valued case, that the order α is given by*

$$\alpha=\lim_{t\to 0^+}\frac{tu'(t;x)}{u(t;x)-x}$$

that corresponds, precisely to [16, Theorem 1] *or* [20, Theorem 9.1].

Now, if A generates the (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ and we consider the Riemann-Liouville in the fractional differential equation

$$\begin{cases} {}^R\partial_t^\alpha u(t) &= Au(t), \quad t>0\\ (g_{1-\alpha}*u)(0) &= x,\end{cases} \tag{3.23}$$

then, its solution is given by

$$u(t;x)=S_{\alpha,\alpha}(t)x.$$

Moreover, we define the operators $\tilde{\varphi}_t : X \to X$ and $\tilde{\psi}_t : X \to X$, respectively, by

$$\tilde{\varphi}_t(x) = \int_0^t u(s;x)ds = (g_1 * u)(t;x) \quad \text{and} \quad \tilde{\psi}_t(x) = u(t;x), \tag{3.24}$$

where $u(t;x)$ is the solution of (3.23).

Now, we have following inverse problem:

- Let $x \in X$ be fixed. Determine $\alpha \in (0,1)$ in (3.23) from the observation data $u(t;x)$ for small t.

The next result gives an answer to this problem for the Riemann-Liouville fractional derivative.

Theorem 3.6. *Suppose that A generates the (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$. Let $x \in X$. If $x^* \in X^*$ with $x^*(x) \neq 0$, then*

$$\alpha = \lim_{t\to 0^+} \frac{x^*(t\tilde{\psi}_t(x))}{x^*(\tilde{\varphi}_t(x))},$$

where $\tilde{\psi}_t$ and $\tilde{\varphi}_t$ are defined in (3.24).

Proof. Let $x \in X$, and $x^* \in X^*$ with $x^*(x) \neq 0$. As $u(t;x) = S_{\alpha,\alpha}(t)x$, we have by Proposition 2.2 and Lemma 2.3 that

$$t^{1-\alpha}u(t;x) = \frac{x}{\Gamma(\alpha)} + At^{1-\alpha}(g_\alpha * S_{\alpha,\alpha})(t)x = \frac{x}{\Gamma(\alpha)} + At^{1-\alpha}S_{\alpha,2\alpha}(t)x.$$

By Proposition 2.2 we have

$$\lim_{t\to 0^+} t^{1-\alpha}AS_{\alpha,2\alpha}(t)x = \lim_{t\to 0^+} \frac{AS_{\alpha,\alpha+1}(t)}{g_{2\alpha}(t)}\frac{t^\alpha}{\Gamma(2\alpha)}x = 0,$$

where the limit is in X. Hence,

$$\lim_{t\to 0^+} x^*(t^{1-\alpha}\tilde{\psi}_t(x)) = \lim_{t\to 0^+} x^*(t^{1-\alpha}u(t;x)) = \frac{x^*(x)}{\Gamma(\alpha)}. \tag{3.25}$$

Since $u(t;x) = g_\alpha(t)x + A(g_\alpha * S_{\alpha,\alpha})(t)x$, integrating over $[0,t]$, by Lemma 2.3 and the semigroup law for the functions g_β, we have that

$$(g_1 * u)(t;x) = g_{\alpha+1}(t)x + A(g_1 * g_\alpha * S_{\alpha,\alpha})(t)x = g_{\alpha+1}(t)x + AS_{\alpha,2\alpha+1}(t)x.$$

Therefore, $t^{-\alpha}(g_1 * u)(t;x) = \frac{x}{\Gamma(\alpha+1)} + At^{-\alpha}S_{\alpha,2\alpha+1}(t)x$. By Proposition 2.2 we get

$$\lim_{t\to 0^+} t^{-\alpha}AS_{\alpha,2\alpha+1}(t)x = \lim_{t\to 0^+} \frac{AS_{\alpha,2\alpha+1}(t)}{g_{2\alpha+1}(t)}\frac{t^\alpha}{\Gamma(2\alpha+1)}x = 0.$$

Hence,

$$\lim_{t\to 0^+} x^*(t^{-\alpha}\tilde{\varphi}_t(x)) = \lim_{t\to 0^+} x^*(t^{-\alpha}(g_1 * u)(t;x)) = \frac{x^*(x)}{\Gamma(\alpha+1)}. \tag{3.26}$$

From (3.25) and (3.26) we deduce that

$$\lim_{t\to 0^+} \frac{x^*(t\tilde{\psi}_t(x))}{x^*(\tilde{\varphi}_t(x))} = \lim_{t\to 0^+} \frac{x^*(t^{1-\alpha}\tilde{\psi}_t(x))}{x^*(t^{-\alpha}\tilde{\varphi}_t(x))} = \frac{\frac{x^*(x)}{\Gamma(\alpha)}}{\frac{x^*(x)}{\Gamma(\alpha+1)}} = \alpha.$$

□

Remark 3.7. *In the scalar case, Theorem 3.6 says that the order α is given by $\alpha = \lim_{t\to 0^+} \frac{tu(t;x)}{(g_1*u)(t;x)}$.*

It is worth noting that the recovery of the precise exponent α as $t \to 0^+$ in Theorems 3.4 and 3.6 can be viewed through the Karamata-Feller-type Tauberian theorem (see [8, Sec. 1.7] and [12, XIII, Sec. 1.5]). In fact, as the Laplace transform of the (α,β)-resolvent family takes the form $\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}$, it behavior for $|\lambda| \to \infty$ is equivalent to the time behavior of $S_{\alpha,\beta}(t)$ as $t \to 0^+$.

To provide an intuitive explanation of this idea, let us consider for simplicity the problem (3.15) assuming A is a bounded operator. In this setting, the solution is given by $u(t;x) = S_{\alpha,1}(t)x$. By (3.17), the Laplace transform of $G(t) := u(t;x) - x$ takes the form $\hat{G}(\lambda) = \lambda^{-1}A(\lambda^\alpha I - A)^{-1}$. Since $u'(t;x) = AS_{\alpha,\alpha}(t)x$ by (3.21), standard properties of the Laplace transform imply that the function $F(t) := tu'(t;x)$ satisfies $\hat{F}(\lambda) = -\frac{d}{d\lambda}\left[A(\lambda^\alpha I - A)^{-1}\right]$, which yields:

$$\hat{F}(\lambda) = \alpha\lambda^{\alpha-1}A(\lambda^\alpha I - A)^{-2}.$$

Assuming further that the operator A is invertible, we obtain $\hat{F}(\lambda)\hat{G}(\lambda)^{-1} = \alpha\lambda^\alpha(\lambda^\alpha I - A)^{-1} = \alpha(I - \frac{A}{\lambda^\alpha})^{-1}$, and consequently, $\lim_{|\lambda|\to\infty} \hat{F}(\lambda)\hat{G}(\lambda)^{-1} = \alpha I$. This structural link indicates that the asymptotic behavior of $\hat{F}(\lambda)\hat{G}(\lambda)^{-1}$ as $|\lambda| \to \infty$ mirrors that of the ratio $tu'(t;x)/(u(t;x) - x)$ as $t \to 0^+$.This connection provides a robust conceptual foundation for our reconstruction formulas and points toward future generalizations involving broader classes of non-local memory kernels.

4. Determination of α for small times. The superdiffusion case: $1 < \alpha < 2$.

In this section, we identify the order α (where $1 < \alpha < 2$) of the fractional differential equations $\partial_t^\alpha u(t) = Au(t)$ and ${}^R\partial_t^\alpha u(t) = Au(t)$, for the Caputo and Riemann-Liouville fractional derivatives.

We first consider the equation for the Caputo fractional derivative

$$\begin{cases} \partial_t^\alpha u(t) &= Au(t), \quad t \geq 0 \\ u(0) &= x \\ u'(0) &= y. \end{cases} \tag{4.27}$$

For each $x, y \in X$, we denote by $u(t;x,y)$ the solution to problem (4.27). If $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is the $(\alpha,1)$-resolvent family generated by A, then,

$$u(t;x,y) = S_{\alpha,1}(t)x + (g_1 * S_{\alpha,1})(t)y,$$

see for instance [34]. By Lemma 2.3 we can write

$$u(t;x,y) = S_{\alpha,1}(t)x + S_{\alpha,2}(t)y.$$

Now, we define the operators $\varphi_t : X \to X$ and $\psi_t : X \to X$, respectively, by

$$\varphi_t(x) = u(t;x,y) - x - ty \quad \text{and} \quad \psi_t(x) = u''(t;x,y), \tag{4.28}$$

where $u(t;x,y)$ is the solution of (4.27) for any fixed $y \in X$.

Next, we consider the inverse problem:

- Let $x \in X$ be fixed. Determine the order $\alpha \in (1,2)$ in (4.27) from the observation data $u(t;x,y)$ for small t.

For the Caputo fractional derivative (4.27) we have the following result.

Theorem 4.8. *Let $y \in X$. Suppose that A generates the $(\alpha,1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$. Let $x \in D(A)$. If $x^* \in X^*$ with $x^*(Ax) \neq 0$, then*

$$\alpha(\alpha-1) = \lim_{t\to 0^+} \frac{x^*(t^2\psi_t(x))}{x^*(\varphi_t(x))}. \tag{4.29}$$

Proof. Let $x \in D(A)$ and $x^* \in X^*$ with $x^*(Ax) \neq 0$. As $u(t;x,y) = S_{\alpha,1}(t)x + S_{\alpha,2}(t)y$, by Proposition 2.2 and Lemma 2.3 we can write

$$u(t;x,y) = x + A(g_\alpha * S_{\alpha,1})(t)x + ty + A(g_\alpha * S_{\alpha,2})(t)y = x + AS_{\alpha,\alpha+1}(t)x + ty + AS_{\alpha,\alpha+2}(t)y. \tag{4.30}$$

Proposition 2.2 implies

$$\lim_{t\to 0^+} t^{-\alpha} AS_{\alpha,\alpha+1}(t)x = \lim_{t\to 0^+} \frac{AS_{\alpha,\alpha+1}(t)}{g_{\alpha+1}(t)} \frac{1}{\Gamma(\alpha+1)} Ax = \frac{1}{\Gamma(\alpha+1)} Ax,$$

and

$$\lim_{t\to 0^+} t^{-\alpha} AS_{\alpha,\alpha+2}(t)x = \lim_{t\to 0^+} \frac{AS_{\alpha,\alpha+2}(t)}{g_{\alpha+2}(t)} \frac{t}{\Gamma(\alpha+2)} Ax = 0,$$

where, both limits are in X. Thus,

$$\lim_{t\to 0^+} x^*(t^{-\alpha}(u(t;x,y) - x - ty)) = \frac{1}{\Gamma(\alpha+1)} x^*(Ax),$$

that is,

$$\lim_{t\to 0^+} x^*(t^{-\alpha}\varphi_t(x)) = \frac{1}{\Gamma(\alpha+1)} x^*(Ax), \quad x \in D(A), y \in X. \tag{4.31}$$

By (3.20), $S'_{\alpha,1}(t) = AS_{\alpha,\alpha}(t)$ and $S'_{\alpha,2}(t) = S_{\alpha,1}(t)$, therefore

$$u'(t;x,y) = S'_{\alpha,1}(t)x + S'_{\alpha,2}(t)y = AS_{\alpha,\alpha}(t)x + S_{\alpha,1}(t)y.$$

As $\alpha > 1$, we have for any $\lambda^\alpha \in \rho(A)$,

$$\widehat{S}'_{\alpha,\alpha}(\lambda)x = \lambda \hat{S}_{\alpha,\alpha}(\lambda)x - S_{\alpha,\alpha}(0)x = \lambda^{\alpha-(\alpha-1)}(\lambda^\alpha - A)^{-1}x = \hat{S}_{\alpha,\alpha-1}(\lambda)x. \tag{4.32}$$

Hence,

$$t^{2-\alpha}u''(t;x,y) = t^{2-\alpha}AS'_{\alpha,\alpha}(t)x + t^{2-\alpha}S'_{\alpha,1}(t)y = t^{2-\alpha}AS_{\alpha,\alpha-1}(t)x + t^{2-\alpha}AS_{\alpha,\alpha}(t)y.$$

By Proposition 2.2 we have

$$\lim_{t\to 0^+} t^{2-\alpha} AS_{\alpha,\alpha-1}(t)x = \lim_{t\to 0^+} \frac{AS_{\alpha,\alpha-1}(t)}{g_{\alpha-1}(t)} \frac{1}{\Gamma(\alpha-1)} x = \frac{Ax}{\Gamma(\alpha-1)}$$

and

$$\lim_{t\to 0^+} t^{2-\alpha} AS_{\alpha,\alpha}(t)x = \lim_{t\to 0^+} \frac{AS_{\alpha,\alpha}(t)}{g_\alpha(t)} \frac{t}{\Gamma(\alpha)} x = 0.$$

Therefore,

$$\lim_{t\to 0^+} x^*(t^{2-\alpha}u''(t;x,y)) = \frac{x^*(Ax)}{\Gamma(\alpha-1)},$$

that is,

$$\lim_{t\to 0^+} x^*(t^{2-\alpha}\psi_t(x)) = \frac{x^*(Ax)}{\Gamma(\alpha-1)}. \tag{4.33}$$

As $\frac{1}{\Gamma(\alpha-1)}\Gamma(\alpha+1) = \alpha(\alpha-1)$, using (4.31) and (4.33) we get

$$\lim_{t\to 0^+} \frac{x^*(t^2\psi_t(x))}{x^*(\varphi_t(x))} = \lim_{t\to 0^+} \frac{x^*(t^{2-\alpha}\psi_t(x))}{x^*(t^{-\alpha}\varphi_t(x))} = \frac{\frac{x^*(Ax)}{\Gamma(\alpha-1)}}{\frac{x^*(Ax)}{\Gamma(\alpha+1)}} = \alpha(\alpha-1).$$

□

Remark 4.9. *We notice that, in the scalar case, the order α in Theorem 4.8 belongs to $(1,2)$ and therefore, if L denotes the limit in the right-hand side of (4.29), that is,*

$$L = \lim_{t\to 0+} \frac{t^2u''(t;x,y)}{u(t;x,y) - x - ty},$$

then

$$\alpha = \frac{1+\sqrt{1+4L}}{2}.$$

Now, we consider the problem for the Riemann-Liouville fractional derivative. If $u(t;x,y)$ denotes the solution to the problem

$$\left\{\begin{array}{rcl} {}^R\partial_t^\alpha u(t) &=& Au(t), \quad t\geq 0\\ (g_{2-\alpha}*u)(0) &=& x\\ (g_{2-\alpha}*u)'(0) &=& y, \end{array}\right. \tag{4.34}$$

and A generates the $(\alpha,\alpha-1)$-resolvent family $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$, then (by [34]), we have

$$u(t;x,y)=S_{\alpha,\alpha-1}(t)x+(g_1*S_{\alpha,\alpha-1})(t)y. \tag{4.35}$$

Moreover, by Lemma 2.3 we can write

$$u(t;x,y)=S_{\alpha,\alpha-1}(t)x+S_{\alpha,\alpha}(t)y. \tag{4.36}$$

Moreover, for any fixed $y\in X$, we define the operators $\tilde{\varphi}_t : X\to X$ and $\tilde{\psi}_t : X\to X$, respectively, by

$$\tilde{\varphi}_t(x)=\int_0^t u(s;x,y)ds=(g_1*u)(t;x,y) \text{ and } \tilde{\psi}_t(x)=\int_0^t (t-s)u(s;x,y)ds=(g_2*u)(t;x,y), \tag{4.37}$$

where $u(t;x,y)$ is the solution of (4.34).

Next, we consider the following inverse problem:

- Let $x\in X$ be fixed. Determinate $\alpha\in(1,2)$ in (4.34) from the observation involving $u(t;x,y)$ for small t.

The next theorem gives an answer to this problem.

Theorem 4.10. *Let $y\in X$. Suppose that A generates the $(\alpha,\alpha-1)$-resolvent family $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$. Let $x\in X$. If $x^*\in X^*$ with $x^*(x)\neq 0$, then*

$$\alpha=\lim_{t\to 0^+}\frac{x^*(t\tilde{\varphi}_t(x))}{x^*(\tilde{\psi}_t(x))},$$

where $\tilde{\varphi}_t$ and $\tilde{\psi}_t$ are defined in (4.37).

Proof. Let $x\in X$, and $x^*\in X^*$ with $x^*(x)\neq 0$. As $u(t;x,y)=S_{\alpha,\alpha-1}(t)x+S_{\alpha,\alpha}(t)y$, by Proposition 2.2, Lemma 2.3, and the semigroup law for the functions g_β, we obtain

$$\begin{aligned} (g_1*u)(t;x,y) &= \int_0^t u(s;x,y)ds\\ &= g_\alpha(t)x+A(g_{\alpha+1}*S_{\alpha,\alpha-1})(t)x+g_{\alpha+1}(t)y+A(g_{\alpha+1}*S_{\alpha,\alpha})(t)y\\ &= g_\alpha(t)x+AS_{\alpha,2\alpha}(t)x+g_{\alpha+1}(t)y+AS_{\alpha,2\alpha+1}(t)y. \end{aligned}$$

Hence,

$$t^{1-\alpha}(g_1*u)(t;x,y)=\frac{1}{\Gamma(\alpha)}x+t^{1-\alpha}AS_{\alpha,2\alpha}(t)x+\frac{ty}{\Gamma(\alpha+1)}+t^{1-\alpha}AS_{\alpha,2\alpha+1}(t)y.$$

Proposition 2.2 implies that, in X we have,

$$\lim_{t\to 0^+}t^{1-\alpha}AS_{\alpha,2\alpha}(t)x=\lim_{t\to 0^+}\frac{AS_{\alpha,2\alpha}(t)}{g_{2\alpha}(t)}\frac{t^\alpha}{\Gamma(2\alpha)}x=0,$$

and

$$\lim_{t\to 0^+}t^{1-\alpha}AS_{\alpha,2\alpha+1}(t)y=\lim_{t\to 0^+}\frac{AS_{\alpha,2\alpha+1}(t)}{g_{2\alpha+1}(t)}\frac{t^{\alpha+1}}{\Gamma(2\alpha+1)}y=0.$$

We obtain $\lim_{t\to 0^+}x^*(t^{1-\alpha}(g_1*u)(t;x,y))=\frac{1}{\Gamma(\alpha)}x^*(x)$, which means that

$$\lim_{t\to 0^+}x^*(t^{1-\alpha}\tilde{\varphi}_t(x))=\frac{1}{\Gamma(\alpha)}x^*(x). \tag{4.38}$$

Now, we integrate (4.36) twice to obtain, by Proposition 2.2 and Lemma 2.3, that

$$\begin{aligned}(g_2 * u)(t;x,y) &= \int_0^t (t-s)u(s;x,y)ds\\ &= g_{\alpha+1}(t)x + A(g_{\alpha+2} * S_{\alpha,\alpha-1})(t)x + g_{\alpha+2}(t)y + A(g_{\alpha+2} * S_{\alpha,\alpha})(t)y\\ &= g_{\alpha+1}(t)x + AS_{\alpha,2\alpha+1}(t)x + g_{\alpha+2}(t)y + AS_{\alpha,2\alpha+2}(t)y. \end{aligned} \tag{4.39}$$

Thus

$$t^{-\alpha}(g_2 * u)(t;x,y) = \frac{1}{\Gamma(\alpha+1)}x + t^{-\alpha}AS_{\alpha,2\alpha+1}(t)x + \frac{ty}{\Gamma(\alpha+2)} + t^{-\alpha}AS_{\alpha,2\alpha+2}(t)y.$$

As

$$\lim_{t\to0^+} t^{-\alpha}AS_{\alpha,2\alpha+1}(t)x = \lim_{t\to0^+}\frac{AS_{\alpha,2\alpha+1}(t)}{g_{2\alpha+1}(t)}\frac{t^{\alpha}}{\Gamma(2\alpha+1)}x = 0$$

and

$$\lim_{t\to0^+} t^{-\alpha}AS_{\alpha,2\alpha+2}(t)y = \lim_{t\to0^+}\frac{AS_{\alpha,2\alpha+2}(t)}{g_{2\alpha+2}(t)}\frac{t^{\alpha+1}}{\Gamma(2\alpha+2)}y = 0,$$

(see Proposition 2.2) we conclude that

$$\lim_{t\to0^+} x^*(t^{-\alpha}\tilde{\psi}_t(x)) = \lim_{t\to0^+} x^*(t^{-\alpha}(g_2 * u)(t;x,y)) = \frac{1}{\Gamma(\alpha+1)}x^*(x). \tag{4.40}$$

As $\frac{1}{\Gamma(\alpha)}\Gamma(\alpha+1) = \alpha$, the conclusion follows as in the proof of Theorem 4.8 using (4.38) and (4.40).

□

Remark 4.11. *We notice that, in the scalar case, the order α in Theorem 4.10 is given by*

$$\alpha = \lim_{t\to0^+}\frac{t(g_1 * u)(t;x,y)}{(g_2 * u)(t;x,y)}.$$

5. Examples

In this section, we show how to compute α from the known solution $u(t;x)$ according to Theorems 3.4-4.10. First, we focus on the case where A is a self-adjoint operator, and then we give an example where the operator in the system is not.

Example 5.12.

Let $-A$ be a non-negative and self-adjoint operator on the Hilbert space $X = L^2(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded and open set. If the operator A has a compact resolvent, then $-A$ has a discrete spectrum and its eigenvalues satisfy $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ with $\lim_{n\to\infty}\lambda_n = \infty$.

If ϕ_n denotes the normalized eigenfunction associated with λ_n, then for all $v \in D(A)$ we have $-Av = \sum_{n=1}^{\infty}\lambda_n\langle v,\phi_n\rangle_{L^2(\Omega)}\phi_n$. Now, consider the problem

$$\begin{cases} \partial_t^\alpha u(t,x) = Au(t,x) & t>0,\\ u(0,x) = u_0(x), \end{cases} \tag{5.41}$$

where $x \in \Omega$ and $u_0 \in L^2(\Omega)$. Multiplying both sides of (5.41) by $\phi_n(x)$ and integrating over Ω we obtain that $u_n(t) := \langle u(t,\cdot),\phi_n(\cdot)\rangle_{L^2(\Omega)}$ is a solution of the scalar-valued problem

$$\begin{cases} \partial_t^\alpha u_n(t) = -\lambda_n u_n(t) & t>0,\\ u_n(0) = u_{0,n}, \end{cases} \tag{5.42}$$

where $u_{0,n} = \langle u_0(\cdot),\phi_n(\cdot)\rangle_{L^2(\Omega)}$, for all $n\in\mathbb{N}$. The solution to (5.42) is given by $u_n(t) = E_{\alpha,1}(-\lambda_n t^\alpha)u_{0,n} =: S^n_{\alpha,1}(t)u_{0,n}$, where $\{S^n_{\alpha,1}(t)\}_{t\ge0}$ is the resolvent family generated by $A_n := -\lambda_n$ (this is possible thanks to the fact that A_n is the generator of a C_0-semigroup, see Section 2). For simplicity, we take $\lambda_n = 1$ and $u_{0,n} = 1$. According to Theorem 3.4 and Remark 3.5, the plots in Figure 1 correspond to $\frac{tu_n'(t)}{u_n(t)-u_{0,n}}$ as $t\to0^+$ for different values of α.

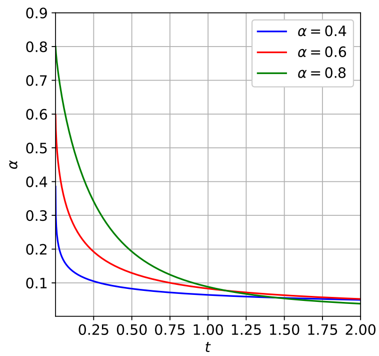

FIGURE 1. Plot of $0 < \alpha < 1$ close to 0.

Now, we consider $1 < \alpha < 2$ and the Problem (1.6) for the operator A described above. For each eigenvalue λ_n, the solution to the corresponding scalar-valued problem is given by (see (1.7))

$$u_n(t) = E_{\alpha,1}(-\lambda_n t^\alpha)u_{0,n} + tE_{\alpha,2}(-\lambda_n t^\alpha)u_{1,n} = S^n_{\alpha,1}(t)u_{0,n} + S^n_{\alpha,2}(t)u_{1,n}, \quad n \in \mathbb{N}.$$

In the next Figure 2, we use Theorem 4.8 and Remark 4.9, to plot $t \mapsto \frac{1+\sqrt{1+4\frac{t^2 u_n''(t)}{u_n(t)-u_{0,n}-tu_{1,n}}}}{2}$ close to $t = 0$ for different choices of $1 < \alpha < 2$. We use here $\lambda_n = 1$ and $u_{0,n} = 1, u_{1,n} = 2$.

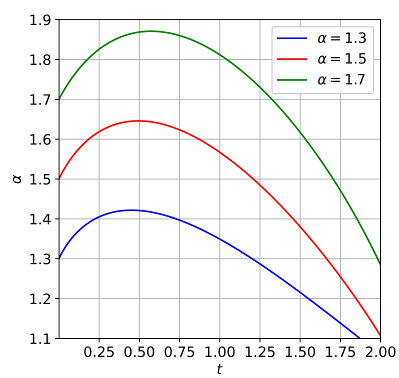

FIGURE 2. Plot of $1 < \alpha < 2$ close to 0.

Now, we consider the fractional differential equations for the Riemann-Liouville fractional derivative (1.8) and (1.9). We first consider $0 < \alpha < 1$. By (1.10), the solution to the corresponding Problem (1.8) is given by

$$u_n(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)u_{0,n} = S^n_{\alpha,\alpha}(t)u_{0,n}, \quad n \in \mathbb{N}.$$

In Figure 3, and according to Theorem 3.6 and Remark 3.7, we plot $t \mapsto \frac{tu_n(t)}{(g_1 * u_n)(t)}$ for small values of t. We take different values of α and $\lambda_n = 1$.

Now, we consider $1 < \alpha < 2$. The solution to the corresponding Problem (1.9) for its eigenvalue is

$$u_n(t) = t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)u_{0,n} + t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)u_{1,n} = S^n_{\alpha,\alpha-1}(t)u_{0,n} + S^n_{\alpha,\alpha}(t)u_{1,n}, \quad n \in \mathbb{N}.$$

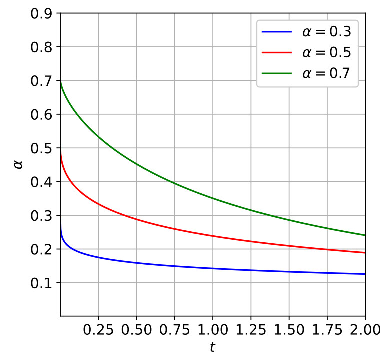

FIGURE 3. Plot of $0 < \alpha < 1$ close to 0.

In the next Figure 4, using Theorem 4.10 and Remark 4.11, we plot $\frac{t(g_1 * u_n)(t)}{(g_2 * u_n)(t)}$ for small values of t. We take different values of α and $\lambda_n = 1$.

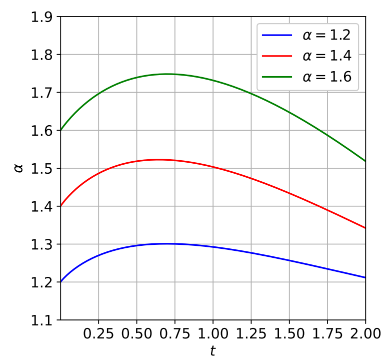

FIGURE 4. Plot of $1 < \alpha < 2$ close to 0.

We emphasize that the spectral setting presented in these examples isolates an specific (but any) eigenvalue of a self-adjoint operator A, which reduces the abstract system to a scalar framework. While this is highly valuable for our reconstruction formulas, these illustrations do not fully capture the complete functional-analytic elements of the abstract operator-theoretic framework developed in Sections 3 and 4.

Example 5.13.

Now, in the space $X = \ell^2(\mathbb{N})$, consider the weighted-shifted operator $A(x_1, x_2, ...) = (0, r_1 x_1, r_2 x_2, ...)$, where $x = (x_k)_{k\in\mathbb{N}} \in X$, and $(r_k)_{k\in\mathbb{N}}$ is a given sequence of real numbers with $|r_k| \leq M$ for all $k \in \mathbb{N}$. Then, A is a linear bounded operator (not self-adjoint) with $\|A\| \leq M$, and therefore, it generates an (α, β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ such that $\hat{S}_{\alpha,\beta}(\lambda) = \lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}$. Since A is a bounded operator, we may choose $|\lambda|$ large enough such that

$$\hat{S}_{\alpha,\beta}(\lambda)x = \lambda^{-\beta}\left(I - \frac{A}{\lambda^\alpha}\right)^{-1} x = \sum_{n=0}^{\infty} \frac{1}{\lambda^{\alpha n+\beta}} A^n x, \quad x \in X.$$

Therefore, the explicit expression to $S_{\alpha,\beta}(t)$ is given by

$$S_{\alpha,\beta}(t)x = \sum_{n=0}^{\infty} g_{\alpha n+\beta}(t)A^n x = \sum_{n=0}^{\infty} \frac{t^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} A^n x, \quad x \in X.$$

Using mathematical induction, we deduce that, for each $x = (x_k)_{k\in\mathbb{N}}$, the sequence $A^n x = ((A^n x)_k)_{k\in\mathbb{N}}$ is given by

$$(A^n x)_k = \begin{cases} (r_{k-1}r_{k-2}...r_{k-n})\, x_{k-n}, & \text{if } 0 \le n \le k-1 \\ 0, & \text{if } n \ge k, \end{cases} \tag{5.43}$$

and therefore, for each $x = (x_k)_{k\in\mathbb{N}} \in X$,

$$(S_{\alpha,\beta}(t)x)_k = g_\beta(t)x_k + g_{\alpha+\beta}(t)(Ax)_k + g_{2\alpha+\beta}(t)(A^2x)_k + ... = \sum_{n=0}^{k-1} g_{\alpha n+\beta}(t)(r_{k-1}r_{k-2}...r_{k-n})x_{k-n}, \quad k \in \mathbb{N}.$$

On the space X, consider the fractional differential equation for the Caputo fractional derivative,

$$\begin{cases} \partial_t^\alpha u(t) &= Au(t) \quad t > 0, \\ u(0) &= x, \end{cases} \tag{5.44}$$

where x is a fixed element in X. Let $u(t;x) = S_{\alpha,1}(t)x$ be the solution to (5.44). According to Theorem 3.4, to find α from the known solution $u(t;x)$, we need to compute the limit

$$\lim_{t\to 0^+} \frac{x^*(tu'(t;x))}{x^*(u(t;x)-x)},$$

for an $x \in D(A)$ and an $x^* \in X^*$ with $x^*(Ax) \neq 0$.

As for any $\gamma > 1$, $g'_\gamma(t) = g_{\gamma-1}(t)$, observe that (with $\beta = 1$),

$$(u(t;x) - x)_k = g_{\alpha+1}(t)(Ax)_k + g_{2\alpha+1}(t)(A^2x)_k + ...$$

and

$$(u'(t;x))_k = g_\alpha(t)(Ax)_k + g_{2\alpha}(t)(A^2x)_k + ...$$

Let $x^* \in \ell^2(\mathbb{N})$. By the Riesz Representation Theorem,

$$x^*(v) = \sum_{k=1}^{\infty} v_k x_k = \langle v, z\rangle_{\ell^2(\mathbb{N})}, \quad v \in \ell^2(\mathbb{N}),$$

where $z = (z_k)_{k\in\mathbb{N}}$ is the sequence representing x^*. Thus, by (5.43)

$$\begin{aligned} x^*(u(t;x)-x) &= \sum_{k=1}^{\infty} z_k(u(t;x)-x)_k \\ &= z_1\left(g_{\alpha+1}(t)(Ax)_1 + g_{2\alpha+1}(t)(A^2x)_1 + ...\right) + z_2\left(g_{\alpha+1}(t)(Ax)_2 + g_{2\alpha+1}(t)(A^2x)_2 + ...\right) + \\ &\vdots \\ &= z_2\left(g_{\alpha+1}(t)r_1x_1\right) + z_3\left(g_{\alpha+1}(t)r_2x_2 + g_{2\alpha+1}(t)r_2r_1x_1\right) + ... \\ &= t^\alpha\left(\frac{1}{\Gamma(\alpha+1)}z_2r_1x_1 + \frac{1}{\Gamma(\alpha+1)}z_3r_2x_2 + \frac{t^\alpha}{\Gamma(2\alpha+1)}z_3r_2r_1x_1 + ...\right). \end{aligned}$$

Similarly,

$$x^*(tu'(t;x)) = \alpha t^\alpha\left(\frac{1}{\Gamma(\alpha+1)}z_2r_1x_1 + \frac{1}{\Gamma(\alpha+1)}z_3r_2x_2 + 2\frac{t^{\alpha+1}}{\Gamma(2\alpha+1)}z_3r_2r_1x_1 + ...\right).$$

Since $Ax = (r_{k-1}x_{k-1})_k$, the condition $x^*(Ax) \neq 0$, is equivalent to

$$\sum_{k=1}^{\infty} z_k r_{k-1} x_{k-1} \neq 0, \tag{5.45}$$

where, by definition, we take $x_0 = 0$. Therefore, if (5.45) holds, then

$$\lim_{t\to 0^+} \frac{x^*(tu'(t;x))}{x^*(u(t;x)-x)} = \alpha.$$

We observe that, if $x = e_1 = (1,0,0,...)$, then $Ax = (0, r_1, 0, ...) = r_1 e_2$. Choosing $x^* \in \ell^2(\mathbb{N})$ represented by $z = e_2$, we get

$$x^*(Ax) = \langle r_1 e_2, e_2 \rangle_{\ell^2(\mathbb{N})} = r_1.$$

Thus, if $r_1 \neq 0$, then the condition (5.45) is easily met.

This example demonstrates that the abstract framework developed in this paper successfully identifies the parameter α across diverse contexts, extending well beyond the classical self-adjoint setting and offering a novel contribution to the literature.

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