ASYMPTOTIC BEHAVIOR OF MILD SOLUTIONS TO FRACTIONAL CAUCHY PROBLEMS IN BANACH SPACES

RODRIGO PONCE

ABSTRACT. In this paper we investigate the existence, asymptotic behavior and uniform p-integrability of fractional resolvent families generated by sectorial operators in Banach spaces. As a consequence, we obtain properties on the behavior of mild solutions to abstract fractional Cauchy problems for the Caputo and Riemann-Liouville fractional derivatives.

1. INTRODUCTION

In this paper is the study the asymptotic behavior of the solutions to the fractional Cauchy problems

$$\partial_t^\alpha u(t) = Au(t) + f(t), t \geq 0, \quad u(0) = x, \quad u'(0) = y, \tag{1.1}$$

and

$$\partial^\alpha u(t) = Au(t) + f(t), t \geq 0, \ (g_{2-\alpha} * u)(0) = x, \ (g_{2-\alpha} * u)'(0) = y, \tag{1.2}$$

where f is a suitable function, A is a closed and linear operator defined in a Banach space X, $x, y \in X$, for $1 < \alpha < 2$, ∂_t^α and ∂^α denote, respectively, the Caputo and Riemann-Liouville fractional derivatives, and for $\mu > 0$, $g_\mu(t) := t^{\mu-1}/\Gamma(\mu)$ (here $\Gamma(\cdot)$ is the Gamma function) and $*$ denotes the usual finite convolution.

Fractional differential equations arise in many areas of applied sciences such as, anomalous diffusion, fractional generalization of the kinetic equation, random walks, fluid flow, rheology, electrical networks, control theory of dynamical systems, viscoelasticity, chemical physics, optics and signal processing, among others, see for instance [5, 8, 11, 17, 23].

As in ordinary differential equations of first or second order, a useful method to solve fractional differential equations is the Laplace transform method, see for instance [18]. This means that if we take Laplace transform in (1.1) and (1.2) then, the *mild* solution to (1.1) and (1.2) are, respectively, given by

$$u(t) = S_{\alpha,1}(t)x + S_{\alpha,2}(t)y + \int_0^t S_{\alpha,\alpha}(t-s)f(s)ds, \quad t \geq 0, \quad \text{and} \tag{1.3}$$

$$u(t) = S_{\alpha,\alpha-1}(t)x + S_{\alpha,\alpha}(t)y + \int_0^t S_{\alpha,\alpha}(t-s)f(s)ds, \quad t \geq 0, \tag{1.4}$$

where, for $\alpha, \beta > 0$, $S_{\alpha,\beta}(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}d\lambda$, $t \geq 0$, and γ is a suitable complex path where the resolvent operator $(\lambda^\alpha - A)^{-1}$ is well defined.

The family of operators $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ it is well-known in some cases. By the uniqueness of the Laplace transform $\{S_{1,1}(t)\}_{t\geq 0}$ corresponds to a C_0-semigroup generated by A, whereas $\{S_{2,1}(t)\}_{t\geq 0}$ and $\{S_{2,2}(t)\}_{t\geq 0}$ are, respectively, the cosine and sine family generated by A, see [3] for further details. If $1 \leq \alpha \leq 2$ and $\beta = 1$, then $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is an α-times resolvent [12]. In this case, $\{S_{\alpha,1}(t)\}_{t\geq 0}$ interpolates between the semigroup ($\alpha = 1$) and the cosine ($\alpha = 2$) case. Thus, if A is the second order operator, then $\{S_{\alpha,1}(t)\}_{t\geq 0}$ interpolates the parabolic problem of first order (heat equation) and the hyperbolic problem of second order (wave equation). Finally, the case $1 \leq \alpha = \beta \leq 2$ corresponds to an α-order resolvent (see [13]) and if $\alpha = 1$ and $\beta = n + 1, n \in \mathbb{N}$, then we get an n-times integrated semigroup, see [3] for more details.

If $A = \rho I$ (where I is the identity operator), $\rho \in \mathbb{C}$, then $S_{\alpha,\beta}(t)$ corresponds to the function $s_{\alpha,\beta}(t) := t^{\beta-1}E_{\alpha,\beta}(\rho t^\alpha)$, where for $\alpha, \beta > 0$, $z \in \mathbb{C}$, $E_{\alpha,\beta}(z)$ is the Mittag-Leffler defined by $E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)}$, whose Laplace transform $\mathcal{L}$, verifies $\mathcal{L}(t^{\beta-1}E_{\alpha,\beta}(\rho t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha-\rho}$, for all $\rho \in \mathbb{C}, \mathrm{Re}\lambda > |\rho|^{1/\alpha}$.

It is a well-known fact that the Mittag-Leffler function arises naturally in the representation of solutions to ordinary fractional differential equations, see for instance [11]. Moreover, the properties of this function $s_{\alpha,\beta}(t)$ (see [10]) are particularly useful to study the properties of solutions to this class of equations. However, in an abstract setting, that is, when A is a closed linear operator defined in a Banach space, many properties of $S_{\alpha,\beta}(t)$ (for $\alpha, \beta > 0$) remain as a not addressed subject in the literature.

The existence of mild solutions to fractional differential equations has been widely studied in the last years, see for instance [2, 4, 5, 8, 12, 21, 22].

2010 *Mathematics Subject Classification.* 33E12; 34A08; 45N05; 47D06; 34K30.

Key words and phrases. Mittag-Leffler function; sectorial operators; asymptotic behavior; mild solution; fractional derivative.

The variation of constants formulas (1.3) and (1.4) suggest that if we know, for example, asymptotic (or integrability) properties of $t \mapsto S_{\alpha,\beta}(t)$ then, we could obtain some asymptotic (or integrability) properties of the solution u to problems (1.1) and (1.2), respectively. If A is a ω-sectorial operator defined in a Banach space X (see its Definition in Section 2) and $1<\alpha<2$, then the mild solution to the problem $v'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Av(s) + f(t)$, $t \geq 0$, under the initial condition $v(0) = v_0 \in X$, is given by $v(t) = S_{\alpha,1}(t)v_0 + \int_0^t S_{\alpha,1}(t-s)f(s)ds$, $t \geq 0$, and there exists $C > 0$ such that $\|S_{\alpha,1}(t)\| \leq C/(1+|\omega|t^{\alpha})$ for all $t \geq 0$, which implies that $\|S_{\alpha,1}(t)\| \to 0$ as $t \to \infty$, see [6, 7]. This asymptotic behavior of $S_{\alpha,1}(t)$ provides several tools to obtain interesting consequences on the solutions to some fractional (and integral) differential equations. See for instance [2, 19, 20, 21, 24] for further details. On the other hand, the asymptotic behavior of $S_{\alpha,1}(t)$ and $S_{\alpha,\alpha}(t)$ for $0<\alpha<1$, has been recently studied in [1, 22]. These works treat the asymptotic behavior for sectorial and almost sectorial operators A, and as consequence, the authors obtain several properties on the solution to the Caputo fractional Cauchy problem in case $0<\alpha<1$.

However, the asymptotic behavior of $S_{\alpha,1}(t)$ (for $1<\alpha<2$) does not allows to obtain asymptotic properties of the solutions u to problems (1.1) and (1.2), because the variation of constants formulas (1.3) and (1.4) involve the function $S_{\alpha,\beta}(t)$ for $\beta \neq 1$. To the best of our knowledge, the asymptotic behavior of $S_{\alpha,\beta}(t)$ is an untreated topic in the existing literature on fractional differential equations in Banach spaces.

In this paper, we study the asymptotic behavior and uniform integrability of $t \mapsto S_{\alpha,\beta}(t)$, for sectorial operators A, where $1<\alpha<2$, and $\beta \geq 1$ are such that $\alpha-\beta+1>0$. As consequence, we obtain several results on the properties of the solutions to the fractional Cauchy problems (1.1) and (1.2). We remark that, we study simultaneously the case of the Riemann-Liouville and Caputo fractional derivatives.

The paper is organized as follows. Section 2 provides the Preliminaries. Section 3 is devoted to a generation theorem and to the asymptotic behavior and p-integrability of $S_{\alpha,\beta}(t)$. Finally, the Section 4 presents results on the regularity of solutions to Problems (1.1) and (1.2). More concretely, we study conditions on α, β and f ensuring that the mild solution u to (1.1) (and (1.2)) belongs to $C_0(\mathbb{R}_+, X)$ or $L^p(\mathbb{R}_+, X)$ (for $1<p<\infty$).

2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. We denote by $\mathcal{B}(X)$ the space of all bounded linear operators from X into X. If A is a closed linear operator on X, we denote by $\rho(A)$ the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ to its resolvent operator, which is defined for all $\lambda \in \rho(A)$.

A strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be exponentially bounded if there exist $M>0$ and $w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{wt}$, for all $t \geq 0$. Moreover, $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called uniformly p-integrable, $1 \leq p < \infty$, if $\|S\|_p := \left(\int_0^\infty \|S(t)\|^p dt\right)^{1/p} < \infty$. For $1 \leq p < \infty$, $L^p(\mathbb{R}_+, X)$ denotes the space of all Bochner measurable functions $g : \mathbb{R}_+ \to X$ such that $\|g\|_p := \left(\int_0^\infty \|g(t)\|^p dt\right)^{1/p} < \infty$.

We say that a closed and densely defined operator A, defined on a Banach space $(X, \|\cdot\|)$, is said to be *ω-sectorial of angle ϕ*, if there exist $\phi \in [0, \pi/2)$ and $\omega \in \mathbb{R}$ such that its resolvent exists in the sector $\omega + \Sigma_\phi := \left\{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(\lambda)| < \frac{\pi}{2} + \phi\right\} \setminus \{\omega\}$ and $\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}$, for all $\lambda \in \omega + \Sigma_\phi$. In case $\omega = 0$ we say that A is sectorial of angle $\phi + \pi/2$. More details on sectorial operators can be found in [9].

For $\mu > 0$, $n := \lceil \mu \rceil$ denotes the smallest integer greater than or equal to μ. The finite convolution of f and g is defined by $(f * g)(t) := \int_0^t f(t-s)g(s)ds$.

Definition 2.1. *Let $\alpha > 0$ and $n = \lceil \alpha \rceil$. The Caputo and Riemann-Liouville fractional derivative of order α of u are defined, respectively, by $\partial_t^\alpha u(t) := \int_0^t g_{n-\alpha}(t-s)u^{(n)}(s)ds$, and $\partial^\alpha u(t) := \frac{d^n}{dt^n} \int_0^t g_{n-\alpha}(t-s)u(s)ds$.*

Denoting by $\hat{f}$ (or $\mathcal{L}(f)$) to the Laplace transform of f, we have for $1<\alpha<2$ that

$$\widehat{\partial_t^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \lambda^{\alpha-1}u(0) - \lambda^{\alpha-2}u'(0), \text{ and } \widehat{\partial^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \lambda(g_{2-\alpha} * u)(0) - (g_{2-\alpha} * u)'(0). \tag{2.1}$$

3. Asymptotic Behavior and Uniform Integrability of the resolvent family

In this section we define a resolvent family of operators generated by an operator A. We also present a generation result and we study the asymptotic behavior and uniform integrability of this family.

Definition 3.2. *Let A be closed linear operator with domain $D(A)$, defined on a Banach space X, $1 \leq \alpha \leq 2$ and $0 < \beta \leq 2$. We say that A is the generator of an (α,β)-resolvent family, if there exists $\omega \geq 0$ and a strongly continuous and exponentially bounded function $S_{\alpha,\beta} : [0,\infty) \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$, and for all $x \in X$,*

$$\lambda^{\alpha-\beta}\left(\lambda^\alpha - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)x\,dt, \quad \mathrm{Re}\lambda > \omega.$$

In this case, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the (α,β)-resolvent family *generated by A.*

Comparing Definition 3.2 with the notion of (a,k)-regularized families in [14] we observe that $t \mapsto S_{\alpha,\beta}(t)$, is a (g_α, g_β)-regularized family. Moreover, the function $S_{\alpha,\beta}(t)$ satisfies the functional equation (see [13, 15]): $S_{\alpha,\beta}(s)(g_\alpha * S_{\alpha,\beta})(t) - (g_\alpha * S_{\alpha,\beta})(s)S_{\alpha,\beta}(t) = g_\beta(s)(g_\alpha * S_{\alpha,\beta})(t) - g_\beta(t)(g_\alpha * S_{\alpha,\beta})(s)$, for all $t, s \geq 0$, and, if an operator A with domain $D(A)$ is the infinitesimal generator of an (α,β)-resolvent family, then for all $x \in D(A)$ we have $Ax = \lim_{t\to 0^+} \frac{S_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)}$. For example, $S_{1,1}(t)$ corresponds to a C_0-semigroup, $S_{2,1}(t)$ to a cosine family and $S_{2,2}(t)$ is a sine family. We notice that in the scalar case, that is, when $A = \rho I$, where $\rho \in \mathbb{C}$ and I denotes the identity operator, then by the uniqueness of the Laplace transform, $S_{\alpha,\beta}(t)$ corresponds to the function $t^{\beta-1}E_{\alpha,\beta}(\rho t^\alpha)$.

We have also the following result. Its proof follows similarly as in [13, Proposition 3.7].

Proposition 3.3. *Let $1 \leq \alpha, \beta \leq 2$. Let $S_{\alpha,\beta}(t)$ be the (α,β)-resolvent family generated by A. Then:*

(1) *$S_{\alpha,\beta}(t)x \in D(A)$ and $S_{\alpha,\beta}(t)Ax = AS_{\alpha,\beta}(t)x$ for all $x \in D(A)$ and $t \geq 0$.*
(2) *If $x \in D(A)$ and $t \geq 0$, then*

$$S_{\alpha,\beta}(t)x = g_\beta(t)x + \int_0^t g_\alpha(t-s)AS_{\alpha,\beta}(s)xds \tag{3.2}$$

(3) *If $x \in X, t \geq 0$, then $\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds \in D(A)$ and $S_{\alpha,\beta}(t)x = g_\beta(t)x + A\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds$. In particular, $S_{\alpha,\beta}(0) = g_\beta(0)I$.*

The next generation result (analogous to the Hille-Yosida Theorem for C_0-semigroups) is contained in [14, Theorem 3.4]. See also [13, Section 3].

Theorem 3.4. *Let A be a closed linear densely defined operator in a Banach space X. Suppose that $1 < \alpha < 2$ and $\beta \geq 1$ such that $\alpha - \beta + 1 > 0$. Then the following assertions are equivalent.*

(1) *The operator A generates an (α,β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ that satisfies $\|S_{\alpha,\beta}(t)\| \leq Me^{\mu t}$ for all $t \geq 0$ and for some constants $M > 0$ and $\mu \in \mathbb{R}$.*
(2) *There exist constants $\mu \in \mathbb{R}$ and $M > 0$ such that $\lambda^\alpha \in \rho(A)$ for all λ with $\mathrm{Re}\lambda > \mu$ and $H(\lambda) := \lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}$ satisfies $\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\lambda-\mu)^{n+1}}$, for all $\mathrm{Re}\lambda > \mu$ and $n \in \mathbb{N}_0$.*

The next result gives sufficient conditions on α, β and A to obtain generators of (α,β)-resolvent families.

Theorem 3.5. *Let $1 < \alpha < 2$ and $\beta \geq 1$ such that $\alpha - \beta + 1 > 0$. Assume that A is ω-sectorial of angle $\frac{(\alpha-1)\pi}{2}$, where $\omega < 0$. Then A generates an exponentially bounded (α,β)-resolvent family.*

Proof. We will show that $\lambda^\alpha \in \rho(A)$ for all λ with $\mathrm{Re}\lambda > 0$, and there exists a constant $C > 0$ such that the function $H(\lambda) := \lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}$, satisfies the estimate $\|\lambda H(\lambda)\| + \|\lambda^2 H'(\lambda)\| \leq C$, for all $\mathrm{Re}\lambda > 0$. In fact, let $h(\lambda) := \lambda^\alpha$ where $\lambda = re^{i\theta}$ with $|\theta| < \frac{\pi}{2}$ and $r > 0$. We notice that

$$\arg(h(re^{i\theta})) = \mathrm{Im}(\ln(h(re^{i\theta}))) = \mathrm{Im}\int_0^\theta \frac{d}{dt}\ln(h(re^{it}))dt = \mathrm{Im}\int_0^\theta \frac{h'(re^{it})ire^{it}}{h(re^{it})}dt.$$

Since $\lambda\frac{h'(\lambda)}{h(\lambda)} = \alpha$, we obtain $\left|\mathrm{Im}\int_0^\theta \frac{h'(re^{it})ire^{it}}{h(re^{it})}dt\right| \leq \int_0^\theta \left|\frac{h'(re^{it})ire^{it}}{h(re^{it})}\right| dt \leq \alpha\theta \leq \frac{(\alpha-1)\pi}{2} + \frac{\pi}{2}$. Therefore, $h(\lambda) \in \Sigma_{\frac{(\alpha-1)\pi}{2}}$ for all $\mathrm{Re}\lambda > 0$, and H is well defined. Since A is ω-sectorial operator, there exists $M > 0$ such that $\|\lambda H(\lambda)\| \leq M\frac{|\lambda|^{\alpha-\beta+1}}{|\lambda^\alpha - \omega|}$, for all $\mathrm{Re}\lambda > 0$. Since $\beta \geq 1$ and $\alpha - \beta + 1 > 0$, we obtain $\|\lambda H(\lambda)\| \leq M$. A simple computation gives $\lambda^2 H'(\lambda) = (\alpha-\beta)\lambda H(\lambda) + \alpha^2\lambda H(\lambda)\lambda^\alpha(\lambda^\alpha - A)^{-1}$, and thus

$$\|\lambda^2 H'(\lambda)\| \leq |\alpha-\beta|\|\lambda H(\lambda)\| + \alpha^2\|\lambda H(\lambda)\lambda^\alpha(\lambda^\alpha - A)^{-1}\| \leq |\alpha-\beta|M + \frac{\alpha^2 M^2|\lambda|^\alpha}{|\lambda^\alpha - \omega|} \leq (|\alpha-\beta| + \alpha^2 M)M.$$

Therefore, $\|\lambda H(\lambda)\| + \|\lambda^2 H'(\lambda)\| \leq M + |\alpha-\beta|M + \alpha^2 M^2$, for all $\mathrm{Re}\lambda > 0$. We conclude by [21, Propositon 0.1] and Theorem 3.4 that the operator A generates an exponentially bounded (α,β)-resolvent family. □

The next Theorem is one of the main results in this paper.

Theorem 3.6. *Let $1 < \alpha < 2$ and $\beta \geq 1$ such that $\alpha - \beta + 1 > 0$. Assume that A is ω-sectorial of angle $\frac{(\alpha-1)}{2}\pi$, where $\omega < 0$. Then, there exists a constant $C > 0$, depending only on α and β, such that*

$$\|S_{\alpha,\beta}(t)\| \leq \frac{Ct^{\beta-1}}{1+|\omega|t^\alpha}, \quad \text{for all } t > 0. \tag{3.3}$$

Proof. Since A is ω-sectorial of angle $\theta := \frac{(\alpha-1)}{2}\pi$ it follows from Theorem 3.5 that $h(\lambda) := \lambda^\alpha \in \rho(A)$ for all $\mathrm{Re}\lambda > 0$, and $\|(\lambda^\alpha - A)^{-1}\| \leq \frac{M}{|\lambda^\alpha - \omega|}$, for all $\lambda \in \mathbb{C}, \mathrm{Re}\lambda > 0$. Next, we write

$$S_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}d\lambda, \tag{3.4}$$

where γ is a positively oriented path lying inside the sector $\omega+\Sigma_\theta$, whose support Γ is the set of $\lambda\in\mathbb{C}$ such that λ^α belongs to the boundary of B_δ, where $B_\delta:=\{\delta+|\omega|+\Sigma_\theta\}+\{\delta+\Sigma_\phi\}$, with $\delta>0$ and $0<\phi<\theta$. With this definition of γ, $(\lambda^\alpha-A)^{-1}$ is well defined and the representation (3.4) of $S_{\alpha,\beta}(t)$ makes sense. We split γ into two part γ_1, γ_2, whose supports Γ_1 and Γ_2 are the sets formed by the complex numbers λ such that λ^α lies on the intersection of Γ and the boundaries of $|\omega|+1/t^\alpha+\Sigma_\theta$ and $1/t^\alpha+\Sigma_\phi$ respectively, i.e.

$$\Gamma_1=\Gamma\cap\overline{\left\{|\omega|+\frac{1}{t^\alpha}+\Sigma_\theta\right\}}\quad\text{and}\quad\Gamma_2=\Gamma\cap\overline{\left\{\frac{1}{t^\alpha}+\Sigma_\phi\right\}}.$$

Therefore, $\Gamma=\Gamma_1\cup\Gamma_2$ and $S_{\alpha,\beta}(t)=I_1(t)+I_2(t)$, for $t\geq 0$, where $I_j(t):=\frac{1}{2\pi i}\int_{\gamma_j}e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha-A)^{-1}d\lambda$, for $j=1,2$. We now estimate the integrals $I_1(t)$ and $I_2(t)$. For the integral $I_1(t)$ we have

$$\|I_1(t)\|\leq\frac{1}{2\pi}\int_{\gamma_1}|e^{\lambda t}||\lambda|^{\alpha-\beta}\|(\lambda^\alpha-A)^{-1}\||d\lambda|\leq\frac{M}{2\pi}\int_{\gamma_1}|e^{\lambda t}|\frac{|\lambda|^{\alpha-\beta}}{|\lambda^\alpha-\omega|}|d\lambda|.$$

Now, we define $\lambda_{\min}$ as the complex $\lambda\in\mathbb{C}$ such that $\mathrm{Im}(\lambda)>0$, and $|\lambda_{\min}^\alpha-\omega|=\mathrm{dist}(L,\omega)$, where L is the line passing by $(|\omega|+1/t^\alpha,0)$ and the intersection of Γ_1 and Γ_2. For $\lambda\in\Gamma_1$ we have that

$$|\lambda_{\min}^\alpha-\omega|\leq|\lambda^\alpha-\omega|\quad\text{and}\quad\cos(\theta)=\sin(\frac{\pi}{2}-\theta)=\frac{|\lambda_{\min}^\alpha-\omega|}{|\omega|+\frac{1}{t^\alpha}}\leq\frac{|\lambda^\alpha-\omega|}{|\omega|+\frac{1}{t^\alpha}}.$$

Therefore, if $\lambda\in\Gamma_1$ then $\frac{1}{|\lambda^\alpha-\omega|}\leq\frac{t^\alpha}{\cos(\theta)(1+|\omega|t^\alpha)}$. Hence,

$$\|I_1(t)\|\leq\frac{Mt^\alpha}{2\pi\cos(\theta)(1+|\omega|t^\alpha)}\int_{\gamma_1}|e^{\lambda t}||\lambda|^{\alpha-\beta}|d\lambda|\leq\frac{Mt^\alpha}{\pi\cos(\theta)(1+|\omega|t^\alpha)}\int_0^\infty e^{-t\cos(\theta)s}s^{\alpha-\beta}ds=\frac{C_\theta t^{\beta-1}}{(1+|\omega|t^\alpha)},$$

where $C_\theta=\frac{M\Gamma(\alpha-\beta+1)}{\pi(\cos(\theta))^{\alpha-\beta+2}}$. Similarly, if $\lambda\in\Gamma_2$, then $\frac{1}{|\lambda^\alpha-\omega|}\leq\frac{t^\alpha}{\cos(\phi)(1+|\omega|t^\alpha)}$, $t\geq 0$, and therefore

$$\|I_2(t)\|\leq\frac{Mt^\alpha}{2\pi\cos(\phi)(1+|\omega|t^\alpha)}\int_{\gamma_1}|e^{\lambda t}||\lambda|^{\alpha-\beta}|d\lambda|\leq\frac{Mt^\alpha}{\pi\cos(\phi)(1+|\omega|t^\alpha)}\int_0^\infty e^{-t\cos(\phi)s}s^{\alpha-\beta}ds=\frac{C_\phi t^{\beta-1}}{(1+|\omega|t^\alpha)},$$

where $C_\phi:=\frac{M\Gamma(\alpha-\beta+1)}{\pi(\cos(\phi))^{\alpha-\beta+2}}$. Therefore, there exists a constant $C>0$, depending only on α and β, such that $\|S_{\alpha,\beta}(t)\|\leq C\frac{t^{\beta-1}}{(1+|\omega|t^\alpha)}$ for all $t\geq 0$. □

Definition 3.7. *The family* $\{S(t)\}_{t\geq 0}\subset\mathcal{B}(X)$ *is called* asymptotically stable *if* $\|S(t)\|\to 0$ *as* $t\to\infty$.

Corollary 3.8. *Let* $(X,\|\cdot\|)$ *be a Banach space. If* $1<\alpha<2$ *and* $\beta\geq 1$ *are such that* $\alpha-\beta+1>0$, *and* A *is* ω*-sectorial operator of angle* $\theta=\frac{(\alpha-1)}{2}\pi$, *where* $\omega<0$, *then* $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ *is asymptotically stable.*

Proof. It follows from (3.3) in Theorem 3.6. □

Corollary 3.9. *Let* $(X,\|\cdot\|)$ *be a Banach space. If* $1\leq\beta<\alpha<2$ *and* A *is* ω*-sectorial of angle* $\theta=\frac{(\alpha-1)}{2}\pi$, *where* $\omega<0$, *then* $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ *is uniformly* $1-$*integrable.*

Proof. By Theorem 3.6 there exists a constant $C>0$, depending only on α and β, such that $\|S_{\alpha,\beta}(t)\|\leq\frac{Ct^{\beta-1}}{(1+|\omega|t^\alpha)}$ for all $t\geq 0$. Therefore, if $B(\cdot,\cdot)$ denotes de Beta function, then we obtain $\int_0^\infty\|S_{\alpha,\beta}(t)\|dt\leq\int_0^\infty\frac{Ct^{\beta-1}}{1+|\omega|t^\alpha}dt=\frac{C}{\alpha}|\omega|^{-\beta/\alpha}B\left(\frac{\beta}{\alpha},1-\frac{\beta}{\alpha}\right)<\infty$. □

4. Asymptotic behavior of mild solutions

We recall that $C_0(\mathbb{R}_+,X)$ is the space of all continuous functions $g:\mathbb{R}_+\to X$ such that $\lim_{t\to\infty}\|g(t)\|=0$. The following results show that the solutions to the fractional Cauchy problem (for the Caputo and Riemann-Liouville derivatives) belong to $C_0(\mathbb{R}_+,X)$.

Let A be a closed linear operator defined in X, $x,y\in X$, and $1<\alpha<2$. First, we consider the initial value problem for the Caputo fractional derivative

$$\left\{\begin{array}{rcl}\partial_t^\alpha u(t)&=&Au(t)+f(t),\quad t\geq 0\\ u(0)&=&x\\ u'(0)&=&y,\end{array}\right.\tag{4.5}$$

Taking Laplace transform in (4.5) we obtain by (2.1) that

$$u(t)=S_{\alpha,1}(t)x+S_{\alpha,2}(t)y+\int_0^t S_{\alpha,\alpha}(t-s)f(s)ds,\quad t\geq 0.\tag{4.6}$$

By a *mild solution* to problem (4.5) we understand a function $u:[0,\infty)\to X$ satisfying (4.6). Similarly, for the Riemann-Liouville fractional derivative, if we take Laplace in the problem

$$\left\{\begin{array}{rcl}\partial^\alpha u(t)&=&Au(t)+f(t),\quad t\geq 0\\ (g_{2-\alpha}*u)(0)&=&x\\ (g_{2-\alpha}*u)'(0)&=&y,\end{array}\right.\tag{4.7}$$

then

$$u(t)=S_{\alpha,\alpha-1}(t)x+S_{\alpha,\alpha}(t)y+\int_0^t S_{\alpha,\alpha}(t-s)f(s)ds, \quad t\geq 0. \tag{4.8}$$

And, analogously, a *mild solution* to problem (4.7) is a function $u:[0,\infty)\to X$ satisfying (4.8).

The following results give some asymptotic properties of the solutions to problems (4.5) and (4.7).

Proposition 4.10. *Let $1<\alpha<2$, $\omega<0$ and A is ω-sectorial of angle $\theta=\frac{(\alpha-1)}{2}\pi$ defined on a Banach space $(X,\|\cdot\|)$. If $f\in L^1(\mathbb{R}_+,X)$ and $y\in X$, then the mild solution u to problem*

$$\begin{cases} \partial^\alpha u(t) &= Au(t)+f(t), \quad t\geq 0\\ (g_{2-\alpha}*u)(0) &= 0\\ (g_{2-\alpha}*u)'(0) &= y, \end{cases} \tag{4.9}$$

belongs to $C_0(\mathbb{R}_+,X)$.

Proof. The mild solution to problem (4.9) is given by

$$u(t)=S_{\alpha,\alpha}(t)y+\int_0^t S_{\alpha,\alpha}(t-s)f(s)ds, \quad t\geq 0. \tag{4.10}$$

The Theorems 3.4 and 3.5 show that the family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ is strongly continuous. By Corollary 3.8 and [3, Proposition 1.3.5] it follows that the convolution $S_{\alpha,\alpha}*f$ belongs to $C_0(\mathbb{R}_+,X)$. Therefore, by Corollary 3.8, $\|u(t)\|\leq\|S_{\alpha,\alpha}(t)\|\,\|y\|+\|(S_{\alpha,\alpha}*f)(t)\|\to 0$, as $t\to\infty$. ☐

We remark that the problem (4.9) has been widely studied in the last years, see for instance [13, 16] and the references therein. In the next result, we consider a non-zero vector in the first initial condition.

Proposition 4.11. *Let $1<\alpha<2$, $\omega<0$ and A is ω-sectorial of angle $\theta=\frac{(\alpha-1)}{2}\pi$ defined on a Banach space $(X,\|\cdot\|)$. If $f\in L^1(\mathbb{R}_+,X)$, then the mild solution to problem (4.5) belongs to $C_0(\mathbb{R}_+,X)$.*

Proof. The mild solution to (4.5) is given by $u(t)=S_{\alpha,1}(t)x+S_{\alpha,2}(t)y+\int_0^t S_{\alpha,\alpha}(t-s)f(s)ds$, $t\geq 0$. As in the proof of Proposition 4.10, we have that the convolution $S_{\alpha,\alpha}*f$ belongs to $C_0(\mathbb{R}_+,X)$, and by Corollary 3.8, $\|u(t)\|\leq\|S_{\alpha,1}(t)\|\,\|x\|+\|S_{\alpha,2}(t)\|\,\|y\|+\|(S_{\alpha,\alpha}*f)(t)\|\to 0$, as $t\to\infty$. Therefore, $u\in C_0(\mathbb{R}_+,X)$. ☐

Theorem 4.12. *Let $(X,\|\cdot\|)$ be a Banach space and $1<p<\infty$. Let $1<\alpha<2$, and $\beta\geq 1$ such that $p(\alpha-\beta+1)>1$. If A is ω-sectorial of angle $\theta=\frac{(\alpha-1)}{2}\pi$, with $\omega<0$, then $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is uniformly $p-$integrable. In particular, $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ is uniformly $p-$integrable.*

Proof. By Theorem 3.6 there exists a constant $C>0$ such that $\|S_{\alpha,\beta}(t)\|\leq\frac{Ct^{\beta-1}}{(1+|\omega|t^\alpha)}$ for all $t\geq 0$. The assumptions on α,β and p imply that

$$\int_0^\infty\|S_{\alpha,\beta}(t)\|^pdt\leq\int_0^\infty\frac{C^pt^{(\beta-1)p}}{(1+|\omega|t^\alpha)^p}dt=\frac{C^p}{\alpha}\frac{1}{|\omega|^{(\beta-1)p/\alpha+1/\alpha-1}}B\left(\frac{(\beta-1)p}{\alpha}+\frac{1}{\alpha},p(1-\frac{\beta}{\alpha})+\frac{1}{\alpha}(p-1)\right).$$

☐

In the following results, we obtain L^p-regularity of the solutions to Problem (4.9).

Corollary 4.13. *Let $1<p<\infty$, $1<\alpha<2$, $\omega<0$ and A is ω-sectorial of angle $\theta=\frac{(\alpha-1)}{2}\pi$ defined on a Banach space $(X,\|\cdot\|)$. If $f\in L^q(\mathbb{R}_+,X)$, $1/p+1/q=1$, then the solution u to Problem (4.9) verifies $\|u(t)\|\to 0$, as $t\to\infty$.*

Proof. Since $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ is uniformly p-integrable by Theorem 4.12 and $f\in L^q(\mathbb{R}_+,X)$ we obtain that $S_{\alpha,\alpha}*f\in C_0(\mathbb{R}_+,X)$ (see [3, Proposition 1.3.5]). Since that the solution to problem (4.9) is given by (4.10), the Corollary 3.8 implies that $u\in C_0(\mathbb{R}_+,X)$. ☐

Corollary 4.14. *Let $1<p<\infty$, $1<\alpha<2$, $\omega<0$ and A is ω-sectorial of angle $\theta=\frac{(\alpha-1)}{2}\pi$ defined on a Banach space $(X,\|\cdot\|)$. If $f\in L^1(\mathbb{R}_+,X)$ then the solution u to Problem (4.9) belongs to $L^p(\mathbb{R}_+,X)$.*

Proof. By Young's inequality (see [3, Proposition 1.3.5]) and Theorem 4.12 it follows that $\|S_{\alpha,\alpha}*f\|_p\leq\|f\|_1\left(\int_0^\infty\|S_{\alpha,\alpha}(t)\|^pdt\right)^{1/p}<\infty$, that is, $S_{\alpha,\alpha}*f\in L^p(\mathbb{R}_+,X)$. Since the solution of problem (4.9) is given by (4.10), the Theorem 4.12 implies that $u\in L^p(\mathbb{R}_+,X)$. ☐

As in the previous results, for the Caputo fractional Cauchy problem (4.5) we have the following corollaries.

Corollary 4.15. *Let $1<p<\infty$, $1<\alpha<2$, $\omega<0$ and A is ω-sectorial of angle $\theta=\frac{(\alpha-1)}{2}\pi$ defined on a Banach space $(X,\|\cdot\|)$. If $f\in L^q(\mathbb{R}_+,X)$, $1/p+1/q=1$, then the solution u to Problem (4.5) verifies $\|u(t)\|\to 0$, as $t\to\infty$.*

Corollary 4.16. *Let $1 < p < \infty$, $1 < \alpha < 2$, $\omega < 0$ and A is ω-sectorial of angle $\theta = \frac{(\alpha-1)}{2}\pi$ defined on a Banach space $(X, \|\cdot\|)$. If $f \in L^1(\mathbb{R}_+, X)$ and $p(\alpha-1) > 1$, then the solution u to Problem (4.5) belongs to $L^p(\mathbb{R}_+, X)$.*

Proof. As in the proof of Corollary 4.14, $S_{\alpha,\alpha} * f \in L^p(\mathbb{R}_+, X)$. The assumption $p(\alpha-1) > 1$ implies that $S_{\alpha,1}$ and $S_{\alpha,2}$ belong to $L^p(\mathbb{R}_+, X)$ by Theorem 4.12. Therefore, $u \in L^p(\mathbb{R}_+, X)$. □

Now, consider again the Riemann-Liouville case (see Problem (4.7)). Since $1 < \alpha < 2$, the Theorem 3.6 does not allow us to conclude that $\|S_{\alpha,\alpha-1}(t)\| \to 0$ as $t \to \infty$. However, we can prove the following result.

Proposition 4.17. *Let $\frac{3}{2} < \alpha < 2$, $\omega < 0$ and A is ω-sectorial of angle $\theta = \frac{(\alpha-1)}{2}\pi$ defined on a Banach space $(X, \|\cdot\|)$. If $f \in L^1(\mathbb{R}_+, X)$, then the mild solution u to Problem (4.7) satisfies $\|(g_{\alpha-1} * u)(t)\| \to 0$ as $t \to \infty$.*

Proof. First, observe that for all $\mathrm{Re}\lambda > 0$, we have $\mathcal{L}(g_{\alpha-1} * S_{\alpha,\alpha-1})(\lambda) = \frac{1}{\lambda^{\alpha-1}}\lambda(\lambda^\alpha - A)^{-1} = \lambda^{\alpha-(2\alpha-2)}(\lambda^\alpha - A)^{-1} = \mathcal{L}(S_{\alpha,2\alpha-2})(\lambda)$. By the uniqueness of the Laplace transform we conclude that $(g_{\alpha-1} * S_{\alpha,\alpha-1})(t) = S_{\alpha,2\alpha-2}(t)$. Since $3/2 < \alpha < 2$, we can apply Corollary 3.8 to conclude that $\|(g_{\alpha-1} * S_{\alpha,\alpha-1})(t)\| \to 0$ as $t \to \infty$. Analogously, $\mathcal{L}(g_{\alpha-1} * S_{\alpha,\alpha})(\lambda) = \lambda^{\alpha-(2\alpha-1)}(\lambda^\alpha - A)^{-1}$ and therefore $(g_{\alpha-1} * S_{\alpha,\alpha})(t) = S_{\alpha,2\alpha-1}(t)$. The Corollary 3.8 implies that $(g_{\alpha-1} * S_{\alpha,\alpha})(t) \to 0$ as $t \to \infty$. Finally, the convolution $g_{\alpha-1} * S_{\alpha,\alpha} * f$ belongs to $C_0(\mathbb{R}_+, X)$ by [3, Proposition 1.3.5], and by (4.8), we obtain that $g_{\alpha-1} * u$ goes to 0 as $t \to \infty$. □

The above Proposition says that in order to guarantee the convergence to zero of the solution u of problem (4.7) we need to integrate $(\alpha-1)$-times the function u.

Acknowledgements. The author thanks to the anonymous referees for their carefully reading of the manuscript and for making suggestions which have improved the previous version of this paper.

References

[1] B. de Andrade, A. de Carvalho, P. Carvalho-Neto, P. Marín-Rubio, *Semilinear fractional differential equations: global solutions, critical nonlinearities and comparison results,* Top. Methods in Nonlinear Analysis, **45** (2015), no. 2, 439-468.

[2] D. Araya, C. Lizama, *Almost automorphic mild solutions to fractional differential equations,* Nonlinear Anal. **69** (2008), 3692-3705.

[3] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-Valued Laplace transforms and Cauchy problems.* Monogr. Math., vol. **96**, Birkhäuser, Basel, 2011.

[4] S. Bu, *Mild well-posedness of equations with fractional derivative,* Math. Nachr. **285** (2012), no. 2-3, 202-209.

[5] P. de Carvalho-Neto, G. Planas, *Mild solutions to the time fractional Navier-Stokes equations in* $\mathbb{R}^N$, J. Differential Equations **259** (2015), no. 7, 2948-2980.

[6] E. Cuesta, *Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations,* Discrete Contin. Dyn. Syst. 2007, Dyn. Syst. and Diff. Eqns. Proc. of the 6th AIMS Int. Conference, suppl., 277-285.

[7] E. Cuesta, C. Palencia, *A numerical method for an integro-differential equations with memory in Banach spaces: Qualitative properties*, SIAM J. Numer. Anal. 41, (2003) 1232-1241.

[8] S. Eidelman, A. Kochubei, *Cauchy problem for fractional diffusion equations,* J. Diff. Equations **199** (2004), (2), 211-255.

[9] M. Haase, *The functional calculus for sectorial operators,* Operator Theory: Advances and applications, 169, Birkäuser Verlag, Basel, 2006.

[10] H. Haubold, A. Mathai, R. Saxena. *Mittag-Leffler Functions and Their Applications,* 2011, arXiv:0909.0230.

[11] A. Kilbas, H. Srivastava, J. Trujillo. *Theory and applications of fractional differential equations,* North-Holland Mathematics studies 204, Elsevier Science B.V., Amsterdam, 2006.

[12] M. Li, C. Chen, F. Li, *On fractional powers of generators of fractional resolvent families,* J. Funct. Anal. **259** (2010) 2702-2726.

[13] K. Li, J. Peng, J. Jia, *Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives,* J. Funct. Anal. 263 (2012), no. 2, 476-510.

[14] C. Lizama, *Regularized solutions for abstract Volterra equations,* J. Math. Anal. Appl. **243** (2000), 278-292.

[15] C. Lizama, F. Poblete, *On a functional equation associated with (a,k)-regularized resolvent families,* Abstr. Appl. Anal, vol. 2012, Article ID 495487, 23 pages, 2012. doi:10.1155/2012/495487.

[16] Z. Mei, J. Peng J., Y. Zhang, *An operator theoretical approach to Riemann-Liouville fractional Cauchy problem,* Math. Nachr. 288 (2015), no. **7**, 784-797.

[17] J. Nieto, *Maximum principles for fractional differential equations derived from Mittag-Leffler functions,* Appl. Math. Lett. 23 (2010), **10**, 1248-1251.

[18] I. Podlubny, *The Laplace transform method for linear differential equations of the fractional order,* arXiv:funct-an/9710005.

[19] R. Ponce, *Bounded mild solutions to fractional integro-differential equations in Banach spaces,* Semigroup Forum, 87, (2013), 377-392, DOI 10.1007/s00233-013-9474-y.

[20] R. Ponce, *Hölder continuous solutions for fractional differential equations and maximal regularity,* J. Diff. Equations, **255** (2013), 3284-3304.

[21] J. Prüss, *Evolutionary Integral Equations and Applications.* Monographs Math., **87**, Birkhäuser Verlag, 1993.

[22] R. Wang, D. Chen, T. Xiao, *Abstract fractional Cauchy problems with almost sectorial operators,* J. Diff. Equations 252 (2012), 202-235.

[23] J. Wang, Y. Zhou, *Mittag-Leffler-Ulam stabilities of fractional evolution equations,* Appl. Math. Lett. 25 (2012), **4**, 723-728.

[24] J. Zhao, Y. K. Chang, G. N'Guérékata, *Asymptotic behavior of mild solutions to semilinear fractional differential equations,* J. Optim. Theory Appl. 156 (2013), no. 1, 106-114.

Universidad de Talca, Instituto de Matemática y Física, Casilla 747, Talca-Chile.
E-mail address: rponce@inst-mat.utalca.cl