DISCRETE SUBDIFFUSION EQUATIONS WITH MEMORY.

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Abstract. In this paper, we study a discrete subdiffusion equation with memory. Based on the backward operator and the theory of fractional resolvent families, we find a discrete fractional resolvent sequence which allows to write the solution to this discrete subdiffusion equation as a variation of constant formula.

1. Introduction

The problem of the heat conduction in materials with memory, was firstly studied by Coleman and Gurtin [12] and Gurtin and Pipkin [21], where the authors deduced a differential equation of first order with memory, which can be written in the form of

$$(1.1)\qquad \begin{cases} u'(t)=Au(t)+\displaystyle\int_0^t a(t-s)Au(s)ds+f(t), \ t\geq 0\\ u(0)=x, \end{cases}$$

where A is a closed operator (typically is the second order operator) defined in a Banach space X, the initial condition x belongs to X, a is a locally integrable kernel known as the *heat relaxation function,* and f is a suitable continuous function. Typical choices of kernels a are given by $a(t)=\rho\frac{t^{\mu-1}}{\Gamma(\mu)}e^{-\beta t}$, where $\rho\in\mathbb{R}$, $\beta\geq 0$ and $\mu>0$, see for instance [49]. The existence and uniqueness of solutions to equation (1.1) has been widely studied in the last five decades, see for instance the monographs [17, 20, 49] and the references therein. More concretely, it is well known that if $a\in W^{1,1}(\mathbb{R}_+)$ (for instance for $\mu>1$) and A is the generator of a C_0-semigroup, then the problem (1.1) has a unique solution u, see for instance [17, Chapter VI, Section 7]. But, if $a\notin W^{1,1}(\mathbb{R}_+)$ (for instance for $0<\mu<1$), then the classical theory of C_0-semigroups does not allow to ensure the existence of such solutions. However, if A generates a resolvent family $\{S^a(t)\}_{t\geq 0}$ (see [15, 52]), then there exists a unique mild solution u to (1.1) given by the variation of constants formula

$$(1.2)\qquad u(t)=S^a(t)x+\int_0^t S^a(t-s)f(s)ds,\quad t\geq 0.$$

Here, the Laplace transform $\hat{\cdot}$, of $S^a(t)$ verifies $\widehat{S^a}(\lambda)=\frac{1}{1+\hat{a}(\lambda)}\left(\frac{\lambda}{1+\hat{a}(\lambda)}-A\right)^{-1}$ for all $\lambda\in\mathbb{C}$ such that $\frac{\lambda}{1+\hat{a}(\lambda)}\in\rho(A)$. We notice that if $a(t)=0$ for all $t\geq 0$, (that is, the problem of the heat conduction without memory) then $S^a(t)$ is precisely the C_0-semigroup generated by the operator A.

On the other hand, in the last two decades, fractional calculus have been used in many mathematical models to describe a wide variety of phenomena, including problems in viscoelasticity, signal and image processing, engineering, fractional Brownian motion, fractional stochastic differential equations, economics, epidemiology and among others. See [9, 23, 26, 29, 44, 45, 51] and the references therein. More specifically, the subdiffusion equation

$$(1.3)\qquad \begin{cases} \partial_t^\alpha u(t)=Au(t)+f(t), \ t\geq 0\\ u(0)=x, \end{cases}$$

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where A is a closed linear operator defined in X, $x \in X$, f is a suitable continuous function and, for $0 < \alpha < 1$, $\partial_t^\alpha u$ denotes the Caputo fractional derivative of u of order, has been studied both in abstract and applied settings. The mild solution to (1.3) can be written again as a variation of constant formula:

$$u(t) = S_{\alpha,1}(t)x + \int_0^t S_{\alpha,\alpha}(t-s)f(s)ds, \tag{1.4}$$

where, for $\alpha, \beta > 0$, $S_{\alpha,\beta}(t)$ is the fractional resolvent family generated by A which can be defined as $S_{\alpha,\beta}(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}d\lambda$, $t \geq 0$, where, γ is a suitable complex path where the resolvent operator $(\lambda^\alpha - A)^{-1}$ is well-defined. We notice that the function $S_{\alpha,\beta}(t)$ corresponds precisely to a generalization of the scalar Mittag-Leffler function, which is defined by $E_{\alpha,\beta}(z) := \sum_{k=0}^\infty z^k/\Gamma(\alpha k + \beta) = \frac{1}{2\pi i}\int_{Ha} e^\mu \mu^{\alpha-\beta}(\mu^\alpha - z)^{-1}d\mu$, $\alpha, \beta > 0, z \in \mathbb{C}$, where, Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ counterclockwise.

Several different time discretizations of integro-differential equations with memory terms of convolution type in the form of (1.1) have been considered by many authors in the last decades. For example, the authors in [50] take the operator A as an unbounded positive-definite self-adjoint operator with dense domain in a Hilbert space and the operator, in [43], the authors consider A as closed linear operator in a Banach space satisfying the resolvent estimate $\|(z - A)^{-1}\| \leq M_\delta/(1 + |z|)$, for $z \in \Sigma_\delta := \{z \neq 0, |\arg(z)| < \delta\} \cup \{0\}$ for some $\delta \in (\frac{1}{2}\pi, \pi)$, where M_δ is a positive constant, and the kernel a verifies appropriate conditions. See also [10, 11, 14] for a different approach to the scalar case. A typical kernel satisfying such conditions is $a(t) = \rho e^{-\beta t}$ with $\rho \in \mathbb{R}$ and $\beta \geq 0$, see [43, Section 2]. In the case of the kernel a defined by $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, time discretizations in Banach spaces have been studied, for example, in [42] for $0 < \alpha < 1$ (where A verifies the same resolvent estimate above) and in [13] for $1 < \alpha < 2$ (where A is a sectorial operator). Finally, very recently, in [38] the authors study a time discretization of (1.1) where A is assumed to be the generator of a resolvent family $\{S^a(t)\}_{t\geq 0}$ for the discrete time step $\tau = 1$ via the Poisson transform [35].

In addition, there is a recent and extensive literature on time discretization of fractional differential equations in the form of (1.3). See for instance [39, 40] for a classical point of view. In [5, 6, 16, 18] the authors study scalar fractional differential equations in the form of (1.3). The authors in [27] study discrete maximal regularity of fractional evolution equations for the Caputo and Riemann-Liouville fractional derivatives on Banach spaces with the UMD property. In [36, 37] the authors develop a method based on operator-valued Fourier multipliers for the well possedness of fractional difference equations in Banach spaces. On the other hand, in [2, 24, 35] the authors study the existence of solutions to fractional difference equations (for $0 < \alpha < 1$) in the form of

$$_C\nabla^\alpha u^n = Au^{n+1}, \quad n \in \mathbb{N}, \tag{1.5}$$

with the initial condition $u^0 = u_0 \in X$, where $_C\nabla^\alpha u^n$ is an approximation of the Caputo fractional derivative $\partial_t^\alpha u(t)$ (at time $t = n$). By using a subordination principle and a discretization via the Poisson transform ([35]), the authors define a discrete fractional resolvent family $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$ generated by the operator A, and then the authors proved that the solution to this equation can be written in terms of the resolvent $\{\mathcal{S}_\alpha(n)\}_{n\in\mathbb{N}_0}$. The case $1 < \alpha < 2$ has been recently studied by using similar methods in [4]. We notice that (1.5) corresponds to a time discretization of the fractional differential equation (1.3) given by the Poisson transformation [35] for the discrete time step size $\tau = 1$. Finally, in [47] the author studies time discretization to (1.3) for a time step size $\tau > 0$ and finds interesting connections between $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$, a discrete fractional resolvent sequence $\{S^n_{\alpha,\beta}\}_{n\in\mathbb{N}_0}$ and the solution to discrete fractional differential equations in the form of

$$_C\nabla^\alpha u^n = Au^n + f^n, \quad n \in \mathbb{N}, \tag{1.6}$$

where $_C\nabla^\alpha u^n$ is an approximation of the Caputo fractional derivative $\partial_t^\alpha u(t)$ (at time $t = \tau n$). More concretely, in [47] has been proved that the solution to (1.6) under the initial condition $u^0 = x$, is given by the variation of constant formula $u^n = S^n_{\alpha,1}x + \tau(S_{\alpha,\alpha} \star f)^n, n \in \mathbb{N}$, where, for $\alpha, \beta > 0$, and $n \in \mathbb{N}_0$, the fractional resolvent sequence $\{S^n_{\alpha,\beta}\}_{n\in\mathbb{N}_0}$ is defined by $S^n_{\alpha,\beta} := \int_0^\infty \rho_n^\tau(t) S_{\alpha,\beta}(t)dt$, and for a fixed $\tau > 0$, $\rho_n^\tau(t) := e^{-\frac{t}{\tau}} \left(\frac{t}{\tau}\right)^n \frac{1}{\tau n!}$, $(S_{\alpha,\alpha} \star f)^n = \sum_{j=0}^n S^{n-j}_{\alpha,\alpha} f^j$ and $f^j := \int_0^\infty \rho_j^\tau(t) f(t)dt$.

On the other hand, the subdiffusion equation with memory

$$\begin{cases} \partial_t^\alpha u(t) = Au(t) + \displaystyle\int_0^t \kappa(t-s)Au(s)ds + f(t), \ t \geq 0 \\ u(0) = x, \end{cases} \tag{1.7}$$

where $0 < \alpha < 1$, A is a closed linear operator defined in a Banach space X, $x \in X$ and κ is suitable kernel has been studied recently in [1, 30, 31, 32] and [48]. Again, the function $\kappa(t) = e^{-\rho t}\frac{t^{\mu-1}}{\Gamma(\mu)}$ where $\rho \geq 0$ and $0 < \mu \leq 1$ corresponds to a typical example of such kernels. However, to the best of our knowledge, there is not literature on time discretization of (1.7) for $0 < \alpha < 1$.

In this paper, we study the discrete subdiffusion equation with memory

$$_C\nabla^\alpha u^n = Au^n + \tau \sum_{j=0}^n \kappa^{n-j} A u^j + f^n, \quad n \in \mathbb{N}, \tag{1.8}$$

under the initial condition $u^0 = x$. Observe that this equation corresponds to a time discretization (for a time step size $\tau > 0$) of (1.7) which can be obtained by multiplying the subdiffusion equation with memory (1.7) by $\rho_n^\tau(t)$, and next integrating over $[0,\infty)$ (see Section 2). Based on the theory of fractional resolvent families for linear and closed operators and on the properties of the function $\rho_n^\tau(t)$ for a time step size $\tau > 0$ (known as *Poisson distribution*), in this paper we study the existence and representation of the solutions to problem (1.8). More precisely, we will show that the solution to equation (1.8) can be written as a variation of parameter formula in terms of certain discrete fractional resolvent family similarly to the case of the equation (1.6). We notice that for $\alpha = 1$, $_C\nabla^1 u^n$ corresponds to the backward Euler difference $(u^n - u^{n-1})/\tau$ and therefore the discrete equation with memory (1.8) generalizes the integro-differential equations proposed in [38, 42, 43, 50], and if $\kappa(t) = 0$ for all $t \geq 0$ and $0 < \alpha < 1$, then (1.8) corresponds to a time discretization of the fractional subdiffusion (1.1).

The paper is structured as follows. In Section 2 we recall the definition of resolvent families and we give some preliminaries on continuous and discrete fractional calculus. In Section 3 we study the discrete fractional subdiffusion equation with memory (1.8). Here, by assuming that A is the generator of a resolvent family, we prove that the equation (1.8) under the initial condition $u^0 = x$ has a unique solution, which can be written as a variation of constant formula. Finally, in Section 4, assuming that $A = \varrho I$ for some $\varrho > 0$ or A is a self-adjoint operator on $L^2(\Omega)$ (where $\Omega \subset \mathbb{R}^N$ is a bounded open set) with compact resolvent, we give an explicit representation of solutions to (1.8).

2. Resolvent families and continuous and discrete fractional calculus

For a given a Banach spaces $(X, \|\cdot\|)$, the Banach space of all bounded and linear operators from X into X is denoted by $\mathcal{B}(X)$. If A is a closed linear operator defined in X, then $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ is its resolvent operator, which is defined for all $\lambda \in \rho(A)$.

We say that a family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is *exponentially bounded* if there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

In this case, the Laplace transform of $S(t)$, $\hat{S}(\lambda)x := \int_0^\infty e^{-\lambda t} S(t)x dt$, is well defined for all $\mathrm{Re}\lambda > \omega$.

Given $\alpha > 0$, the function g_α is defined by $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\Gamma(\cdot)$ denotes the Gamma function. We note that if $\alpha, \beta > 0$, then $g_{\alpha+\beta} = g_\alpha * g_\beta$, where $(f * g)$ is the usual finite convolution $(f * g)(t) = \int_0^t f(t-s)g(s)ds$. For a locally integrable function $f : [0,\infty) \to X$, we define the *Laplace transform* of f, denoted by $\hat{f}(\lambda)$ (or $\mathcal{L}(f)(\lambda)$) as

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \mathrm{Re}\lambda > \omega$$

whenever the integral is absolutely convergent for $\mathrm{Re}\lambda > \omega$.

Definition 2.1. *Let A be a closed and linear operator defined in a Banach space X and $a \in L^1_{\rm loc}(\mathbb{R}_+)$. We say that A is the generator of a* resolvent family, *if there exist $M > 0$, $\omega \geq 0$ and a strongly continuous function $S^a : [0,\infty) \to \mathcal{B}(X)$ such that $\|S^a(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, $\{\frac{\lambda}{1+\hat{a}(\lambda)} : {\rm Re}\lambda > \omega\} \subset \rho(A)$ and for all $x \in X$,*

$$\frac{1}{1+\hat{a}(\lambda)}\left(\frac{\lambda}{1+\hat{a}(\lambda)} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S^a(t)x dt, \ \ {\rm Re}\lambda > \omega.$$

In this case, $\{S^a(t)\}_{t\geq 0}$ is called the resolvent family *generated by A.*

Now, we notice that if A is the generator of the resolvent family $\{S^a(t)\}_{t\geq 0}$, and $c(t) := 1$, $b(t) := 1 + (1 * a)(t)$, then $\{S^a(t)\}_{t\geq 0}$ corresponds to a (b, c)-regularized family according to [34]. This implies that if $a \equiv 0$, then $\{S^a(t)\}_{t\geq 0}$ is the C_0-semigroup generated by A. Moreover, it is a well-known fact that if A generates a resolvent family $\{S^a(t)\}_{t\geq 0}$, then solution u to (1.1) is given by the variation of parameters formula (1.2).

Definition 2.2. *Let A be a closed and linear operator defined on a Banach space X and $\kappa \in L^1_{\rm loc}(\mathbb{R}_+)$. Given $\alpha, \beta > 0$ we say that A is the generator of an (α, β)-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S^\kappa_{\alpha,\beta} : (0,\infty) \to \mathcal{B}(X)$ such that $S^\kappa_{\alpha,\beta}(t)$ is exponentially bounded, $\left\{\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)} : {\rm Re}\lambda > \omega\right\} \subset \rho(A)$, and for all $x \in X$,*

$$\frac{\lambda^{\alpha-\beta}}{1+\hat{\kappa}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S^\kappa_{\alpha,\beta}(t)x dt, \ \ {\rm Re}\lambda > \omega. \tag{2.9}$$

In this case, $\{S^\kappa_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the (α, β)-resolvent family *generated by A.*

We observe that if $\alpha = \beta = 1$, then a $(1, 1)$-resolvent family $\{S^\kappa_{1,1}(t)\}_{t\geq 0}$ corresponds to the resolvent family $\{S^\kappa(t)\}_{t\geq 0}$ according to Definition 2.1. Moreover, a closed linear operator A generates a unique (α, β)-resolvent family, and if $c(t) := g_\alpha(t) + (\kappa * g_\alpha)(t)$ and A is the generator of an (α, β)-resolvent family $\{S^\kappa_{\alpha,\beta}(t)\}_{t>0}$ then $\{S^\kappa_{\alpha,\beta}(t)\}_{t>0}$ is a (c, g_β)-regularized family as well (according to [34]), and then we can prove the following result, see [34] for further details. See also [1, Definition 2.3 and Remark 2.4] and [3, Section 4]

Proposition 2.3. *If $\alpha, \beta > 0$ and A generates an (α, β)-resolvent family $\{S^\kappa_{\alpha,\beta}(t)\}_{t>0}$, then*

1. $\displaystyle\lim_{t\to 0^+} \frac{S^\kappa_{\alpha,\beta}(t)x}{g_\beta(t)} = x$, *for all $x \in X$,*
2. $S^\kappa_{\alpha,\beta}(t)x \in D(A)$ *and* $S^\kappa_{\alpha,\beta}(t)Ax = AS^\kappa_{\alpha,\beta}(t)x$ *for all $x \in D(A)$ and $t > 0$*
3. *For all $x \in D(A)$,*
$$S^\kappa_{\alpha,\beta}(t)x = g_\beta(t)x + \int_0^t c(t-s)AS^\kappa_{\alpha,\beta}(s)xds,$$
4. $\int_0^t c(t-s)S^\kappa_{\alpha,\beta}(s)xds \in D(A)$ *and*
$$S^\kappa_{\alpha,\beta}(t)x = g_\beta(t)x + A\int_0^t c(t-s)S^\kappa_{\alpha,\beta}(s)xds,$$
for all $x \in X$,

where $c(t) := g_\alpha(t) + (\kappa * g_\alpha)(t)$.

For $\alpha, \beta > 0$ and $z \in \mathbb{C}$, the Mittag-Leffler function $E_{\alpha,\beta}$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{j=0}^\infty \frac{z^j}{\Gamma(\alpha j + \beta)}.$$

Given $\alpha > -1, \beta \in \mathbb{C}$ and $z \in \mathbb{C}$, the Wright function $W_{\alpha,\beta}$ is defined by

$$W_{\alpha,\beta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{j!\Gamma(\alpha j + \beta)}.$$

If $\beta \geq 0$, then for all $z \in \mathbb{C}$ and $\alpha > -1$, we have (see [41]) that

$$W_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{H_a} \mu^{-\beta} e^{\mu + z\mu^{-\alpha}} d\mu,$$

where H_a denotes the Hankel path defined as a contour that begins and $t = -\infty - ia$ $(a > 0)$, encircles the branch cut that lies along the negative real axis, and ends up at $t = -\infty + ib$ $(b > 0)$, see for instance [41].

Definition 2.4. [3, Definition 3.1] *For* $0 < \alpha < 1$ *and* $\beta \geq 0$, *we define the function* $\psi_{\alpha,\beta}$ *in two variables by*

$$\psi_{\alpha,\beta}(t,s) := t^{\beta-1} W_{-\alpha,\beta}(-st^{\alpha}), \quad t > 0, s \in \mathbb{C}.$$

By [3, Theorem 3.2] it follows that if $0 < \alpha < 1$ and $\beta \geq 0$, then $\psi_{\alpha,\beta}(t,s) \geq 0$ for $t, s > 0$ and

$$\int_0^{\infty} e^{-\lambda t} \psi_{\alpha,\beta}(t,s) dt = \lambda^{-\beta} e^{-\lambda^{\alpha} s}, \text{ for } s, \lambda > 0. \tag{2.10}$$

Moreover, there exists an interesting connection between $S^a(t)$ and $S^{\kappa}_{\alpha,\beta}(t)$. In fact, let $0 < \alpha < 1$ and $\varepsilon \geq 0$, and let $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$ be a given kernel and assume that there exist $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\nu \leq 0$ and such that $\hat{a}(\lambda^{\alpha}) = \hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda) > \nu$. Suppose that A is the generator of a resolvent family $\{S^a(t)\}_{t\geq 0}$. Then, A is also the generator of the $(\alpha, \alpha+\varepsilon)$-resolvent family $\{S^{\kappa}_{\alpha,\alpha+\varepsilon}(t)\}_{t>0}$ defined by

$$S^{\kappa}_{\alpha,\alpha+\varepsilon}(t)x := \int_0^{\infty} \psi_{\alpha,\varepsilon}(t,s) S^a(s)x ds, \quad t > 0, x \in X$$

where $\psi_{\alpha,\varepsilon}$ is the Wright type function given in Definition 2.4. Moreover, if $\varepsilon > 0$, then $S^{\kappa}_{\alpha,\alpha+\varepsilon}(t)x = (g_{\varepsilon} * S^{\kappa}_{\alpha,\alpha})(t)x$, for all $x \in X$ and $t > 0$.

In particular, if we take $\varepsilon = 0$ and $\varepsilon = 1 - \alpha$, then we obtain the following subordination result.

Proposition 2.5. [48] *Let* $0 < \alpha < 1$. *Let* $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$ *be a given kernel. Assume that there exist* $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ *and* $\nu \leq 0$ *such that* $\hat{a}(\lambda^{\alpha}) = \hat{\kappa}(\lambda)$ *for all* $\text{Re}(\lambda) > \nu$. *Suppose that* A *is the generator of a resolvent family* $\{S^a(t)\}_{t\geq 0}$ *such that* $\|S^a(t)\| \leq Me^{\omega t}$ *for all* $t \geq 0$, *where* $M, \omega \geq 0$. *Then,* A *is the generator of the resolvent families* $\{S^{\kappa}_{\alpha,\alpha}(t)\}_{t>0}$ *and* $\{S^{\kappa}_{\alpha,1}(t)\}_{t>0}$ *which are, respectively, defined by*

$$S^{\kappa}_{\alpha,\alpha}(t)x := \int_0^{\infty} \psi_{\alpha,0}(t,s) S^a(s)x ds, \quad t > 0, \tag{2.11}$$

and

$$S^{\kappa}_{\alpha,1}(t)x := \int_0^{\infty} \psi_{\alpha,1-\alpha}(t,s) S^a(s)x ds, \quad t > 0. \tag{2.12}$$

We notice that if $\kappa(t) = 0$ for all $t \geq 0$, then a kernel a satisfying the above conditions is $a(t) = 0$ for all $t \geq 0$. Therefore, if A is the generator of a resolvent family $\{S^a(t)\}_{t\geq 0}$ (with $a \equiv 0$), that is, A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then A also generates the resolvent families

$$S^{\kappa}_{\alpha,\alpha}(t)x := \int_0^{\infty} \psi_{\alpha,0}(t,s) T(s)x ds, \quad \text{and} \quad S^{\kappa}_{\alpha,1}(t)x := \int_0^{\infty} \psi_{\alpha,1-\alpha}(t,s) T(s)x ds, t > 0,$$

These last relations are known as *subordination principles,* see for instance [3, 7, 8, 28].

For $0 < \alpha < 1$, the Caputo fractional derivative of order α of a function f is defined by

$$\partial_t^{\alpha} f(t) := (g_{1-\alpha} * f')(t) = \int_0^t g_{1-\alpha}(t-s) f'(s) ds.$$

It is well known that if $\alpha = 1$, then $\partial_t^1 = \frac{d}{dt}$. For further details on fractional calculus we refer to the reader to [41]. Moreover, an easy computation shows that $\hat{g}_\alpha(\lambda) = \frac{1}{\lambda^\alpha}$ for all $\mathrm{Re}(\lambda) > 0$ and applying the properties of the Laplace transform, we obtain

$$\widehat{\partial_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1} f(0) \tag{2.13}$$

for $0 < \alpha \leq 1$. Here, the power λ^α is uniquely defined by $\lambda^\alpha := |\lambda|^\alpha e^{i\arg(\lambda)}$, with $-\pi < \arg(\lambda) < \pi$.

Now, we review some details on discrete fractional calculus. We refer the reader to [19, 47] for further details. We denote the set of all non-negative integers by $\mathbb{N}_0$ and the non-negative real numbers by $\mathbb{R}_0^+$. Give $\tau > 0$ fixed and $n \in \mathbb{N}_0$, we define

$$\rho_n^\tau(t) := e^{-\frac{t}{\tau}} \left(\frac{t}{\tau}\right)^n \frac{1}{\tau n!}.$$

An easy computation shows that $\rho_n^\tau(t) \geq 0$, $\rho_n^\tau(t) = \tau^{-1}\rho_n(t/\tau)$ where $\rho_n(t) := e^{-t}t^n/n!$, and

$$\int_0^\infty \rho_n^\tau(t)dt = 1, \quad \text{for all} \quad n \in \mathbb{N}_0.$$

For a bounded and locally integrable function $u : \mathbb{R}_0^+ \to X$, we define the sequence $(u^n)_n$ (known as *Poisson transformation*, see [35]) by

$$u^n := \int_0^\infty \rho_n^\tau(t)u(t)dt, \quad n \in \mathbb{N}_0.$$

We observe that for $\tau > 0$ small enough, the function $\rho_n^\tau(t)$ behaves like a delta function at $t_n := \tau n$ and then, u^n corresponds to an approximation of u at t_n.

Given the Banach space X, $\mathcal{F}(\mathbb{R}_0^+; X)$ denotes the vector space of all vector-valued functions $v : \mathbb{R}_0^+ \to X$. The *backward Euler operator* $\nabla_\tau : \mathcal{F}(\mathbb{R}_0^+; X) \to \mathcal{F}(\mathbb{R}_0^+; X)$ is defined by

$$\nabla_\tau v^n := \frac{v^n - v^{n-1}}{\tau}, \quad n \in \mathbb{N}.$$

For $m \geq 2$, we define recursively $\nabla_\tau^m : \mathcal{F}(\mathbb{R}_0^+; X) \to \mathcal{F}(\mathbb{R}_0^+; X)$ as

$$\nabla_\tau^m v^n := \begin{cases} \nabla_\tau^{m-1}(\nabla_\tau v)^n, & n \geq m \\ 0, & n < m, \end{cases} \tag{2.14}$$

where $\nabla_\tau^1 \equiv \nabla_\tau$ and ∇_τ^0 is the identity operator. The operator ∇_τ^m is called the *backward difference operator of order* m. It is easy to see that if $v \in \mathcal{F}(\mathbb{R}_0^+; X)$, then

$$(\nabla_\tau^m v)^n = \frac{1}{\tau^m} \sum_{j=0}^m \binom{m}{j} (-1)^j v^{n-j}, \quad n \in \mathbb{N}.$$

Now, we define the sequence

$$k_\tau^\alpha(n) := \tau \int_0^\infty \rho_n^\tau(t) g_\alpha(t)dt, \quad n \in \mathbb{N}_0, \alpha > 0. \tag{2.15}$$

An easy computation shows that

$$k_\tau^\alpha(n) = \frac{\tau^\alpha \Gamma(\alpha + n)}{\Gamma(\alpha)\Gamma(n+1)} = \tau \frac{\Gamma(\alpha + n)}{\Gamma(n+1)} g_\alpha(\tau), \quad n \in \mathbb{N}_0, \alpha > 0.$$

Definition 2.6. *Let $0 < \alpha < 1$. The $\alpha^{\text{th}}-$fractional sum of $v \in \mathcal{F}(\mathbb{R}; X)$ is defined by*

$$(\nabla_\tau^{-\alpha} v)^n := \sum_{j=0}^n k_\tau^\alpha(n-j)v^j, \quad n \in \mathbb{N}_0. \tag{2.16}$$

Definition 2.7. *Let $0 < \alpha < 1$. The* Caputo fractional backward difference operator of order α of v, *${}_C\nabla^\alpha : \mathcal{F}(\mathbb{R}_+; X) \to \mathcal{F}(\mathbb{R}_+; X)$, is defined by*

$$({}_C\nabla^\alpha v)^n := \nabla_\tau^{-(1-\alpha)}(\nabla_\tau^1 v)^n, \quad n \in \mathbb{N}.$$

As in [19, Chapter 1, Section 1.5] we define by convention $\sum_{j=0}^{-k} v^j = 0$, for all $k \in \mathbb{N}$.

If $\alpha = 1$, then the fractional backward difference operator ${}_C\nabla^\alpha$ is defined as the backward difference operator ∇_τ. From [47] we have that if $0 < \alpha < 1$ and $n \in \mathbb{N}$, then ${}_C\nabla^{\alpha+1} v^n = {}_C\nabla^\alpha(\nabla^1 v)^n$, and moreover, we have the following result that relates the Caputo fractional derivative and the Caputo difference operator.

Proposition 2.8. *Let $0 < \alpha < 1$. If $u : [0, \infty) \to X$ is differentiable and bounded, then $\int_0^\infty \rho_n^\tau(t)\partial_t^\alpha u(t)dt = {}_C\nabla^\alpha u^n$, for all $n \in \mathbb{N}$.*

Thus, ${}_C\nabla^\alpha v^n$, corresponds to an approximation of the Caputo fractional derivative $\partial_t^\alpha u(t)$ at the point $t_n = n\tau$.

Now, given a family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$, we define the sequence

$$S^n x := \int_0^\infty \rho_n^\tau(t)S(t)x dt, \quad n \in \mathbb{N}_0, x \in X.$$

Similarly, if $c : \mathbb{R}_+ \to \mathbb{C}$ is a continuous and bounded function, we define $c^n := \int_0^\infty \rho_n^\tau(t)c(t)dt, n \in \mathbb{N}_0$, and the discrete convolution is defined by

$$(c \star S)^n := \sum_{k=0}^{n} c^{n-k} S^k, \quad n \in \mathbb{N}_0.$$

The next result summarizes several properties of the sequences defined above. We refer the reader to [35] and [47] for further details.

Proposition 2.9. *Let $\tau > 0$ be fixed. Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1/\tau)$ exists.*

(1) *If $c : \mathbb{R}_+ \to \mathbb{C}$ is Laplace transformable such that $\hat{c}(1/\tau)$ exists, then*

$$\int_0^\infty \rho_n^\tau(t)(c * S)(t)x dt = \tau(c \star S)^n x, \quad n \in \mathbb{N}_0, \text{ for all } x \in X.$$

(2) *If $0 < \alpha < 1$, then*

$$\int_0^\infty \rho_n^\tau(t)(g_\alpha * S)(t)x dt = \sum_{j=0}^{n} k_\tau^\alpha(n-j)S^j x, \quad n \in \mathbb{N}_0, \text{ for all } x \in X.$$

(3) *If $f : \mathbb{R}_+ \to X$ is Laplace transformable such that $\hat{f}(1/\tau)$ exists, then*

$$\int_0^\infty \rho_n^\tau(t)(S * f)(t)x dt = \tau(S \star f)^n x = \tau \sum_{j=0}^{n} S^{n-j} f^j, \quad n \in \mathbb{N}_0.$$

Finally, we have the following Lemma.

Lemma 2.10. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a family of exponentially bounded linear operators such that $\hat{S}(1/\tau)$ exists. If $f : \mathbb{R}_+ \to X$, $a : \mathbb{R}_+ \to \mathbb{C}$, and $\hat{a}(1/\tau)$ and $\hat{f}(1/\tau)$ exist, then*

$$\tau^2(a \star S \star f)^n = \int_0^\infty \rho_n^\tau(t)(a * S * f)(t)dt,$$

for all $n \in \mathbb{N}_0$, where $(a \star S \star f)^n := (a \star (S \star f))^n$. Moreover, $(a \star (S \star f))^n = ((a \star S) \star f)^n$ for all $n \in \mathbb{N}_0$.

Proof. Since $(a * S * f)(t) = (a * (S * f))(t)$ for all $t \geq 0$, the Proposition 2.9 and the definition of discrete convolution imply that

$$\int_0^\infty \rho_n^\tau(t)(a * S * f)(t)dt = \tau(a \star (S * f))^n = \tau \sum_{k=0}^{n} a^{n-k}(S * f)^k = \tau^2 \sum_{k=0}^{n} a^{n-k}(S \star f)^k = \tau^2(a \star (S \star f))^n,$$

for all $n \in \mathbb{N}_0$. □

3. Solutions to a discrete fractional differential equation with memory

Now, for $0<\alpha<1$, we consider the equation

$$ {}_C\nabla^{\alpha}u^n = Au^n + \tau\sum_{j=0}^{n}\kappa^{n-j}Au^j + f^n, \quad n\in\mathbb{N}, \tag{3.17}$$

under the initial condition $u^0=x$. The main result in this section is the following theorem.

Theorem 3.11. *Let $\tau>0$ and $0<\alpha<1$. Let A be the generator of an (α,α)-resolvent family $\{S^{\kappa}_{\alpha,\alpha}(t)\}_{t\geq 0}$ exponentially bounded with $\|S_{\alpha,\alpha}(t)\|\leq Me^{\omega t}$. If $x\in X$ and f is bounded, then the fractional difference equation* (3.17) *under the initial condition $u^0=x$ has a unique solution given by*

$$u^n = S^{\kappa,n}_{\alpha,1}x + \tau(S^{\kappa}_{\alpha,\alpha}\star f)^n, \tag{3.18}$$

for all $n\in\mathbb{N}$, where $S^{\kappa}_{\alpha,1}(t):=(g_{1-\alpha}*S^{\kappa}_{\alpha,\alpha})(t)$ and

$$S^{\kappa,n}_{\alpha,1} := \int_0^{\infty}\rho^{\tau}_n(t)S^{\kappa}_{\alpha,1}(t)dt.$$

Proof. As in the proof of [35, Theorem 4.4] it is easy to see that $S^{\kappa,n}_{\alpha,1}x\in D(A)$ for all $n\in\mathbb{N}_0$ and $x\in X$. From Proposition 2.3 we know that

$$S^{\kappa}_{\alpha,1}(t)x = x + A\int_0^t c(t-s)S^{\kappa}_{\alpha,1}(s)xds = x + A(c*S^{\kappa}_{\alpha,1})(t)x,$$

for all $t\geq 0$ and $x\in X$, where $c(t)=g_{\alpha}(t)+(\kappa*g_{\alpha})(t)$. Multiplying this equality by $\rho^{\tau}_j(t)$ and integrating over $[0,\infty)$ we conclude (by Proposition 2.9) that

$$S^{\kappa,j}_{\alpha,1}x = x + \tau A\sum_{l=0}^{j}c^{j-l}S^{\kappa,l}_{\alpha,1}x, \tag{3.19}$$

for all $j\geq 0$ and $x\in X$. Now, for all $n\in\mathbb{N}$ we have by definition that

$${}_C\nabla^{\alpha}(S^{\kappa}_{\alpha,1}x)^n = \nabla^{-(1-\alpha)}_{\tau}\nabla^1_{\tau}(S^{\kappa}_{\alpha,1}x)^n = \sum_{j=0}^{n}k^{1-\alpha}_{\tau}(n-j)(\nabla^1_{\tau}S^{\kappa}_{\alpha,1}x)^j,$$

and by (3.19) we get

$$(\nabla^1_{\tau}S^{\kappa}_{\alpha,1}x)^j = \frac{1}{\tau}(S^{\kappa,j}_{\alpha,1}x - S^{\kappa,j-1}_{\alpha,1}x) = A\sum_{l=0}^{j}c^{j-l}S^{\kappa,l}_{\alpha,1}x - A\sum_{l=0}^{j-1}c^{j-1-l}S^{\kappa,l}_{\alpha,1}x$$

for all $j\geq 1$. Let $R(t):=(c*S^{\kappa}_{\alpha,1})(t)$. By Proposition 2.9 we have

$$R^j = \tau\sum_{l=0}^{j}c^{j-l}S^{\kappa,l}_{\alpha,1},$$

which implies that

$$\sum_{j=0}^{n}k^{1-\alpha}_{\tau}(n-j)\sum_{l=0}^{j}c^{j-l}S^{\kappa,l}_{\alpha,1}x = \frac{1}{\tau}\sum_{j=0}^{n}k^{1-\alpha}_{\tau}(n-j)R^jx = \frac{1}{\tau}\int_0^{\infty}\rho^{\tau}_n(t)(g_{1-\alpha}*R)(t)xdt.$$

Since $c(t)=g_{\alpha}(t)+(\kappa*g_{\alpha})$ and $(g_{\alpha}*g_{1-\alpha})(t)=g_1(t)$, we have by definition of R that

$$(g_{1-\alpha}*R)(t) = (g_{1-\alpha}*c*S^{\kappa}_{\alpha,1})(t) = (g_1*S^{\kappa}_{\alpha,1})(t) + (g_1*\kappa*S^{\kappa}_{\alpha,1})(t),$$

and then, the Proposition 2.9 implies again that

$$\begin{aligned}\int_0^\infty \rho_n^\tau(t)(g_{1-\alpha}*R)(t)xdt &= \int_0^\infty \rho_n^\tau(t)(g_1*S^\kappa_{\alpha,1})(t)xdt+\int_0^\infty \rho_n^\tau(t)(g_1*\kappa*S^\kappa_{\alpha,1})(t)xdt\\ &= \sum_{j=0}^n k^1_\tau(n-j)S^{\kappa,j}_{\alpha,1}x+\sum_{j=0}^n k^1_\tau(n-j)(\kappa*S^\kappa_{\alpha,1})^jx.\end{aligned}$$

Since $k^1_\tau(n)=\tau$ for all $n\in\mathbb{N}$, and by Proposition 2.9

$$(\kappa*S^\kappa_{\alpha,1})^jx=\int_0^\infty \rho_j^\tau(t)(\kappa*S^\kappa_{\alpha,1})(t)xdt=\tau\sum_{l=0}^j\kappa^{j-l}S^{\kappa,l}_{\alpha,1}x,$$

we conclude that

$$\sum_{j=0}^n k^{1-\alpha}_\tau(n-j)\sum_{l=0}^j c^{j-l}S^{\kappa,l}_{\alpha,1}x=\sum_{j=0}^n S^{\kappa,j}_{\alpha,1}x+\tau\sum_{j=0}^n\sum_{l=0}^j\kappa^{j-l}S^{\kappa,l}_{\alpha,1}x.$$

Since $\sum_{j=0}^{-l}v^j=0$ for all $l\in\mathbb{N}$, we can prove similarly that

$$\sum_{j=0}^n k^{1-\alpha}_\tau(n-j)\sum_{l=0}^{j-1} c^{j-1-l}S^{\kappa,l}_{\alpha,1}x=\sum_{j=0}^{n-1} S^{\kappa,j}_{\alpha,1}x+\tau\sum_{j=0}^{n-1}\sum_{l=0}^j\kappa^{j-l}S^{\kappa,l}_{\alpha,1}x.$$

Hence,

$$\begin{aligned}{}_C\nabla^\alpha(S^\kappa_{\alpha,1}x)^n &= A\sum_{j=0}^n k^{1-\alpha}_\tau(n-j)\sum_{l=0}^j c^{j-l}S^{\kappa,l}_{\alpha,1}x-A\sum_{j=1}^n k^{1-\alpha}_\tau(n-j)\sum_{l=0}^{j-1}c^{j-1-l}S^{\kappa,l}_{\alpha,1}x\\ &= A\sum_{j=0}^n S^{\kappa,j}_{\alpha,1}x-A\sum_{j=0}^{n-1}S^{\kappa,j}_{\alpha,1}x+\tau A\left[\sum_{j=0}^n\sum_{l=0}^j\kappa^{j-l}S^{\kappa,l}_{\alpha,1}x-\sum_{j=0}^{n-1}\sum_{l=0}^j\kappa^{j-l}S^{\kappa,l}_{\alpha,1}x\right]\\ &= AS^{\kappa,n}_{\alpha,1}x+\tau A\sum_{j=0}^n\kappa^{n-j}S^{\kappa,j}_{\alpha,1}x\\ &= AS^{\kappa,n}_{\alpha,1}x+\tau A(\kappa\star S^\kappa_{\alpha,1})^nx,\end{aligned}$$

for all $n\in\mathbb{N}$ and $x\in X$. Therefore

(3.20) $$_C\nabla^\alpha(S^\kappa_{\alpha,1})^nx=AS^{\kappa,n}_{\alpha,1}x+\tau A(\kappa\star S^\kappa_{\alpha,1})^nx.$$

On the other hand,

$$\begin{aligned}{}_C\nabla^\alpha((S^\kappa_{\alpha,\alpha}\star f)^n) &= \nabla_\tau^{-(1-\alpha)}\nabla^1_\tau(S^\kappa_{\alpha,\alpha}\star f)^n\\ &= \sum_{j=0}^n k^{1-\alpha}_\tau(n-j)\nabla^1_\tau(S^\kappa_{\alpha,\alpha}\star f)^j\\ &= \frac{1}{\tau}\sum_{j=0}^n k^{1-\alpha}_\tau(n-j)(S^\kappa_{\alpha,\alpha}\star f)^j-\frac{1}{\tau}\sum_{j=1}^n k^{1-\alpha}_\tau(n-j)(S^\kappa_{\alpha,\alpha}\star f)^{j-1}.\end{aligned}$$

By Proposition 2.9 we deduce that

(3.21) $$(S_{\alpha,\alpha}\star f)^j=\frac{1}{\tau}(S_{\alpha,\alpha}*f)^j,$$

and, for all $t\geq 0$ and $x\in X$ we have, by Proposition 2.3, that

$$S^\kappa_{\alpha,\alpha}(t)x=g_\alpha(t)x+A(c*S^\kappa_{\alpha,\alpha})(t)x=g_\alpha(t)x+A(g_\alpha*S^\kappa_{\alpha,\alpha})(t)x+A(g_\alpha*\kappa*S^\kappa_{\alpha,\alpha})(t)x.$$

Hence

$$(S^\kappa_{\alpha,\alpha}*f)(t)=(g_\alpha*f)(t)+A(g_\alpha*S^\kappa_{\alpha,\alpha}*f)(t)+A(g_\alpha*\kappa*S^\kappa_{\alpha,\alpha}*f)(t).$$

Multiplying this equality by $\rho_j^\tau(t)$ and integrating over $[0,\infty)$ we get

$$(S^\kappa_{\alpha,\alpha} * f)^j = (g_\alpha * f)^j + A(g_\alpha * S^\kappa_{\alpha,\alpha} * f)^j + A(g_\alpha * \kappa * S^\kappa_{\alpha,\alpha} * f)^j.$$

By Proposition 2.9, Lemma 2.10 and equation (3.21), this last equality is equivalent to

$$\tau(S^\kappa_{\alpha,\alpha} \star f)^j = \sum_{l=0}^{j} k_\tau^\alpha(j-l) f^l + \tau A \sum_{l=0}^{j} k_\tau^\alpha(j-l)(S^\kappa_{\alpha,\alpha} \star f)^l + \tau^2 A \sum_{l=0}^{j} k_\tau^\alpha(j-l)(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^l.$$

Hence,

$$\begin{aligned}
{}_C\nabla^\alpha((S^\kappa_{\alpha,\alpha} \star f)^n) &= \frac{1}{\tau}\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\Bigg[\frac{1}{\tau}\sum_{l=0}^{j} k_\tau^\alpha(j-l) f^l + A\sum_{l=0}^{j} k_\tau^\alpha(j-l)(S^\kappa_{\alpha,\alpha} \star f)^l \\
&\quad + \tau A\sum_{l=0}^{j} k_\tau^\alpha(j-l)(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^l\Bigg] \\
&- \frac{1}{\tau}\sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j)\Bigg[\frac{1}{\tau}\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l) f^l + A\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)(S^\kappa_{\alpha,\alpha} \star f)^l \\
&\quad + \tau A\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^l\Bigg].
\end{aligned}$$

As before, we can prove that

$$\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^\alpha(j-l) f^l = \tau\sum_{j=0}^{n} f^j, \quad \sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l) f^l = \tau\sum_{j=0}^{n-1} f^j,$$

$$\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^\alpha(j-l)(S^\kappa_{\alpha,\alpha} \star f)^l = \tau\sum_{j=0}^{n}(S^\kappa_{\alpha,\alpha} \star f)^l,$$

$$\sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)(S^\kappa_{\alpha,\alpha} \star f)^l x = \tau\sum_{j=0}^{n-1}(S^\kappa_{\alpha,\alpha} \star f)^l,$$

$$\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^\alpha(j-l)(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^l = \tau\sum_{j=0}^{n}(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^l,$$

and

$$\sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^l x = \tau\sum_{j=0}^{n-1}(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^l.$$

Hence,

$$\begin{aligned}
{}_C\nabla^\alpha((S_{\alpha,\alpha} \star f)^n) &= \frac{1}{\tau}\left[\sum_{j=0}^{n} f^j - \sum_{j=0}^{n-1} f^j\right] + A\left[\sum_{j=0}^{n}(S^\kappa_{\alpha,\alpha} \star f)^l - \sum_{j=0}^{n-1}(S^\kappa_{\alpha,\alpha} \star f)^l\right] \\
&\quad + \tau A\left[\sum_{j=0}^{n}(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^l - \sum_{j=0}^{n-1}(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^l\right] \\
&= \frac{1}{\tau} f^n + A(S^\kappa_{\alpha,\alpha} \star f)^n + \tau A(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^n,
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$ {}_C\nabla^\alpha(\tau(S_{\alpha,\alpha} \star f)^n) = f^n + \tau A(S^\kappa_{\alpha,\alpha} \star f)^n + \tau^2 A(\kappa \star S^\kappa_{\alpha,\alpha} \star f)^n, \tag{3.22}$$

for all $n \in \mathbb{N}$.

By (3.20) and (3.22) we conclude that if $u^n := S^{\kappa,n}_{\alpha,1}x + \tau(S^{\kappa}_{\alpha,\alpha} \star f)^n$, then

$$\begin{aligned} {}_C\nabla^{\alpha}(u^n) &= {}_C\nabla^{\alpha}\left(S^{\kappa,n}_{\alpha,1}x + \tau(S^{\kappa}_{\alpha,\alpha} \star f)^n\right) \\ &= AS^{\kappa,n}_{\alpha,1}x + \tau A(\kappa \star S^{\kappa}_{\alpha,1})^n x + \tau A(S^{\kappa}_{\alpha,\alpha} \star f)^n + \tau^2 A(\kappa \star S^{\kappa}_{\alpha,\alpha} \star f)^n + f^n \\ &= A\left[S^{\kappa,n}_{\alpha,1}x + \tau(\kappa \star S^{\kappa}_{\alpha,1})^n x + \tau(S^{\kappa}_{\alpha,\alpha} \star f)^n + \tau^2(\kappa \star S^{\kappa}_{\alpha,\alpha} \star f)^n\right] + f^n. \end{aligned}$$

This implies that

$$ {}_C\nabla^{\alpha}(u^n) = Au^n + A\left[\tau(\kappa \star S^{\kappa}_{\alpha,1})^n + \tau^2(\kappa \star S^{\kappa}_{\alpha,\alpha} \star f)^n\right] + f^n. \tag{3.23}$$

Now, we notice that

$$\begin{aligned} \tau\sum_{j=0}^{n}\kappa^{n-j}Au^j &= \tau A\sum_{j=0}^{n}\kappa^{n-j}\left[S^{\kappa,j}_{\alpha,1}x + \tau(S^{\kappa}_{\alpha,\alpha} \star f)^j\right] \\ &= \tau A(\kappa \star S^{\kappa}_{\alpha,1})^n x + \tau^2 A\sum_{j=0}^{n}\kappa^{n-j}(S^{\kappa}_{\alpha,\alpha} \star f)^j \\ &= \tau A(\kappa \star S^{\kappa}_{\alpha,1})^n x + \tau^2 A(\kappa \star S^{\kappa}_{\alpha,\alpha} \star f)^n. \end{aligned}$$

Replacing this last equality in (3.23) we conclude that

$$ {}_C\nabla^{\alpha}(u^n) = Au^n + \tau\sum_{j=0}^{n}\kappa^{n-j}Au^j + f^n,$$

which means that u^n solves the equation (3.17). The uniqueness, follows from the uniqueness of the resolvent family $\{S^{\kappa}_{\alpha,\alpha}(t)\}_{t\geq 0}$ generated by A. □

In the next result we use the subordination principle given in Proposition 2.5.

Theorem 3.12. *Let $\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be a given kernel. Assume that there exist $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\nu \leq 0$ such that $\hat{a}(\lambda^{\alpha}) = \hat{\kappa}(\lambda)$ for all $\mathrm{Re}(\lambda) > \nu$. Suppose that A is the generator of a resolvent family $\{S^a(t)\}_{t\geq 0}$ such that $\|S^a(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, where $M, \omega \geq 0$. Then, the solution to (3.17) under the initial condition $u^0 = x$, is given by*

$$u^n = S^{\kappa,n}_{\alpha,1}x + \tau(S^{\kappa}_{\alpha,\alpha} \star f)^n,$$

where

$$S^{\kappa,n}_{\alpha,1} = \int_0^{\infty}\int_0^{\infty}\rho^{\tau}_n(t)\psi_{\alpha,1-\alpha}(t,s)S^a(s)dsdt \quad \text{and} \quad S^{\kappa,n}_{\alpha,\alpha} = \int_0^{\infty}\int_0^{\infty}\rho^{\tau}_n(t)\psi_{\alpha,0}(t,s)S^a(s)dsdt. \tag{3.24}$$

Proof. By Proposition 2.5, the operator A generates the resolvent families $\{S^{\kappa}_{\alpha,1}(t)\}_{t>0}$ and $\{S^{\kappa}_{\alpha,\alpha}(t)\}_{t>0}$ defined, respectively, by (2.11) and (2.12). Hence,

$$S^{\kappa,n}_{\alpha,1}x = \int_0^{\infty}\rho^{\tau}_n(t)S^{\kappa}_{\alpha,1}(t)xdt = \int_0^{\infty}\int_0^{\infty}\rho^{\tau}_n(t)\psi_{\alpha,1-\alpha}(t,s)S^a(s)xdsdt,$$

for all $n \in \mathbb{N}_0$, and $x \in X$. Analogously,

$$S^{\kappa,n}_{\alpha,\alpha} = \int_0^{\infty}\int_0^{\infty}\rho^{\tau}_n(t)\psi_{\alpha,0}(t,s)S^a(s)xdsdt.$$

Therefore, the result follows from Theorem 3.11. □

Remark 3.13. *Observe that if $\kappa(t) = 0$ for all $t \geq 0$, then $a(t) = 0$ satisfies the condition in Theorem 3.12 and therefore $\{S^a(t)\}_{t\geq 0}$ corresponds to the C_0-semigroup generated by A. Thus, by* [2, Theorem 3.5] *the operator A generates a discrete α-resolvent family according to* [2, Definition 3.1] *which coincides with the discrete resolvent family $\{S^{\kappa,n}_{\alpha,\alpha}\}_{n\in\mathbb{N}_0}$ defined in* (3.24).

Corollary 3.14. *Let $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$ be a given kernel. Assume that there exist $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\nu \le 0$ such that $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda) > \nu$. Suppose that A is the generator of a resolvent family $\{S^a(t)\}_{t\ge 0}$ such that $\|S^a(t)\| \le Me^{\omega t}$ for all $t \ge 0$, where $M, \omega \ge 0$. Then, the sequences $\{S^{\kappa,n}_{\alpha,1}\}_{n\in\mathbb{N}_0}$ and $\{S^{\kappa,n}_{\alpha,\alpha}\}_{n\in\mathbb{N}_0}$ can be written as*

$$S^{\kappa,n}_{\alpha,1} = \frac{1}{\tau^{n+1}}\int_0^\infty r^{\alpha n}E^{n+1}_{\alpha,\alpha n+1}\left(-\frac{r^\alpha}{\tau}\right)S^a(r)dr$$

and

$$S^{\kappa,n}_{\alpha,\alpha} = \frac{1}{\tau^{n+1}}\int_0^\infty r^{\alpha(n+1)-1}E^{n+1}_{\alpha,\alpha(n+1)}\left(-\frac{r^\alpha}{\tau}\right)S^a(r)dr,$$

where, for $p, q, r > 0$, $E^r_{p,q}(z)$ is the generalized Mittag-Leffler function defined by

$$E^r_{p,q}(z) = \sum_{j=0}^\infty \frac{(r)_j z^j}{j!\Gamma(pj+q)}, \quad z \in \mathbb{C},$$

where $(r)_j$ denotes the Pochhammer symbol defined by $(r)_j := \frac{\Gamma(r+j)}{\Gamma(r)}$.

Proof. In fact, by Theorem 3.12 and Fubini's theorem, we have

$$S^{\kappa,n}_{\alpha,1} = \int_0^\infty\int_0^\infty \rho^\tau_n(t)\psi_{\alpha,1-\alpha}(t,r)S^a(r)dsdt = \int_0^\infty\int_0^\infty \rho^\tau_n(t)\psi_{\alpha,1-\alpha}(t,r)dtS^a(r)dr.$$

Now, by [1, Proposition 2.1], we have

$$\begin{aligned}\int_0^\infty \rho^\tau_n(t)\psi_{\alpha,1-\alpha}(t,r)dt &= \int_0^\infty e^{-\frac{1}{\tau}t}\left(\frac{t}{\tau}\right)^n\frac{1}{\tau n!}\psi_{\alpha,1-\alpha}(t,r)dt\\ &= \frac{1}{\tau^{n+1}}\int_0^\infty e^{-\frac{1}{\tau}t}g_{n+1}(t)\psi_{\alpha,1-\alpha}(t,r)dt\\ &= \frac{1}{\tau^{n+1}}r^{\alpha n}E^{n+1}_{\alpha,\alpha n+1}\left(-\frac{1}{\tau}r^\alpha\right).\end{aligned}$$

And, similarly,

$$\int_0^\infty \rho^\tau_n(t)\psi_{\alpha,0}(t,r)dt = \frac{1}{\tau^{n+1}}r^{\alpha(n+1)-1}E^{n+1}_{\alpha,\alpha(n+1)}\left(-\frac{1}{\tau}r^\alpha\right).$$

□

Remark 3.15. *By* [33, Formula (3.8) in Corollay 3.3] *and* [33, Corollay 3.3 (b)], *we notice that the Wright type functions $\psi_{\alpha,0}$ and $\psi_{\alpha,1-\alpha}$ in Theorem 3.12 can be written, respectively as*

$$\psi_{\alpha,0}(t,s) = \frac{1}{\pi}\int_0^\infty e^{t\rho\cos\theta - s\rho^\alpha\cos\alpha\theta}\cdot\sin(t\rho\sin\theta - s\rho\sin\alpha\theta + \theta)d\rho,$$

for $\pi/2 < \theta < \pi$ and

$$\psi_{\alpha,1-\alpha}(t,s) = \frac{1}{\pi}\int_0^\infty \rho^{\alpha-1}e^{-s\rho^\alpha\cos\alpha(\pi-\theta)-t\rho\cos\theta}\cdot\sin\left(t\rho\sin\theta - s\rho^\alpha\sin\alpha(\pi-\theta) + \alpha(\pi-\theta)\right)d\rho,$$

for $\theta \in (\pi - \frac{\pi}{2\alpha}, \pi/2)$.

In Theorem 3.12 and Corollary 3.14, we need to assume that the operator A is the generator of a resolvent family $\{S^a(t)\}_{t\ge0}$. In the following result, which is a direct consequence of [48, Theorem 3], we study such conditions. We recall that a linear operator $A : D(A) \subset X \to X$ is said to be *ω-sectorial of angle θ* if there are constants $\omega \in \mathbb{R}$, $M > 0$ and $\theta \in (\pi/2, \pi)$ such that $\rho(A) \supset \Sigma_{\theta,\omega} := \{z \in \mathbb{C} : z \ne \omega : |\arg(z-\omega)| < \theta\}$ and $\|(z-A)^{-1}\| \le M/|z-\omega|$ for all $\in \Sigma_{\theta,\omega}$. If A is 0-sectorial of angle θ, we write $A \in \text{Sect}(\theta, M)$. These operators have been studied widely, both in abstract settings (see for instance [22]) and for their applications in the study of linear and nonlinear integro/differential equations, see for example [13, 28, 46, 53].

On the other hand, a kernel $b \in L^1_{\text{loc}}(\mathbb{R}_+)$ is called 1-*regular* (of constant c) if there is a constant $c > 0$ such that

$$|\lambda \hat{b}'(\lambda)| \le c|\hat{b}(\lambda)|, \quad \text{for all} \quad \text{Re}(\lambda) > 0, \tag{3.25}$$

where $\hat{b}'(\lambda)$ is the derivative of $\hat{b}(\lambda)$ with respect to λ. For further details and properties on regular kernels, we refer the reader to [49, Chapter I, Section 3].

Proposition 3.16. *Let $A \in \text{Sect}(\theta, M)$ be a sectorial operator, $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\frac{1}{2} < \alpha < 1$. Assume that there exist $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\nu \le 0$ such that $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda) > \nu$. Suppose that κ is a 1-regular kernel and that the constant c in (3.25) satisfies $(1 + \frac{c}{\alpha})\frac{\pi}{2} \le \theta$. Then, the problem (3.17) under the initial condition $u^0 = x$, has a unique solution.*

Proof. By [48, Theorem 3], the operator A generates a resolvent family $\{S^a(t)\}_{t\ge 0}$. And, by Theorem 3.12 the equation (3.17) has a unique solution, which is given by (3.18), where the sequences $\{S^{\kappa,n}_{\alpha,1}\}_{n\in\mathbb{N}_0}$ and $\{S^{\kappa,n}_{\alpha,\alpha}\}_{n\in\mathbb{N}_0}$ are given in (3.24). □

4. Examples

Suppose that $A = \varrho I$ for some $\varrho > 0$, and assume that $\rho, \mu > 0$, $\gamma \in \mathbb{R} \setminus \{0\}$ and $\frac{1}{2} < \alpha < 1$. Let $\kappa(t) = \gamma \frac{t^{\mu-1}}{\Gamma(\mu)} e^{-\rho t}$. Assume that f is a bounded function. Since

$$\begin{aligned} \kappa^n &= \int_0^\infty \rho_n^\tau(t)\kappa(t)dt \\ &= \gamma \int_0^\infty e^{-\frac{t}{\tau}} \left(\frac{t}{\tau}\right)^n \frac{1}{\tau n!} \frac{t^{\mu-1}}{\Gamma(\mu)} e^{-\rho t} dt \\ &= \frac{\gamma}{\tau^{\mu+n}} \frac{\Gamma(n+\mu)}{\Gamma(\mu)\Gamma(n+1)} \int_0^\infty e^{-(\frac{1}{\tau}+\rho)t} g_{n+\mu}(t)dt \\ &= \frac{\gamma}{\tau} k^\mu_\tau(n) \frac{1}{(1+\rho\tau)^{n+\mu}}, \end{aligned}$$

where $k^\mu_\tau(n)$ is the sequence defined in (2.15), the homogeneous discrete subdiffusion problem (3.17) under the initial condition $u^0 = x$ reads

$$_C\nabla^\alpha u^n = \varrho u^n + \gamma \varrho \sum_{j=0}^n k^\mu_\tau(n-j) \frac{1}{(1+\rho\tau)^{n-j+\mu}} u^j + f^n, \quad n \in \mathbb{N}. \tag{4.26}$$

Since the Laplace transform of $\hat{\kappa}(\lambda) = \frac{\gamma}{(\lambda+\rho)^\mu}$, for all $\lambda > -\rho$, the kernel a in Theorem 3.12 satisfying $\hat{a}(\lambda^\alpha) = \gamma/(\lambda+\rho)^\mu$, is given by (see for instance [25, Formula (11.13)])

$$a(t) = \gamma t^{\frac{\mu}{\alpha}-1} E^\mu_{\frac{1}{\alpha},\frac{\mu}{\alpha}}(-\rho t^{\frac{1}{\alpha}}), \quad t \ge 0. \tag{4.27}$$

If $0 < \mu < \frac{1}{2}$, then by [48, Section 3], the operator A generates the resolvent family $\{S^a(t)\}_{t\ge 0}$ given by

$$S^a(t) = \sum_{i=0}^\infty \frac{\varrho^i \gamma^i}{i!} \int_0^t (t-s)^i e^{\varrho(t-s)} s^{\frac{\mu i}{\alpha}-1} E^{\mu i}_{\frac{1}{\alpha},\frac{\mu i}{\alpha}}(-\rho s^{\frac{1}{\alpha}}) ds \tag{4.28}$$

and, the solution to the problem

$$\begin{cases} u'(t) = \varrho u(t) + \varrho \displaystyle\int_0^t a(t-s)u(s)ds + f(t), \quad t > 0, \\ u(0) = x, \end{cases}$$

is given by $u(t) = S^a(t)x + (S^a * f)(t)$. Moreover, we have the following result.

Proposition 4.17. *Suppose that $\varrho > 0$, $\gamma \in \mathbb{R}$ and $\frac{1}{2} < \alpha < 1$. If $0 < \mu < \frac{1}{2}$, then the unique solution u to the scalar Problem (4.26) under the initial condition $u^0 = x$ is given by*

(4.29)
$$u^n=\sum_{i=0}^{\infty}\frac{\varrho^i\gamma^i}{i!}\int_0^\infty\int_0^\infty\rho_n^\tau(t)\psi_{\alpha,1-\alpha}(t,s)\int_0^s(s-r)^ie^{\varrho(s-r)}r^{\frac{\mu i}{\alpha}-1}E^{\mu i}_{\frac{1}{\alpha},\frac{\mu i}{\alpha}}(-\rho r^{\frac{1}{\alpha}})xdrdsdt$$
$$+\tau\sum_{j=0}^{n}\sum_{i=0}^{\infty}\frac{\varrho^i\gamma^i}{i!}\int_0^\infty\int_0^\infty\rho_n^\tau(t)\psi_{\alpha,0}(t,s)\int_0^s(s-r)^ie^{\varrho(s-r)}r^{\frac{\mu i}{\alpha}-1}E^{\mu i}_{\frac{1}{\alpha},\frac{\mu i}{\alpha}}(-\rho r^{\frac{1}{\alpha}})f^jdrdsdt,\quad n\in\mathbb{N},$$

where $\psi_{\alpha,1-\alpha},\psi_{\alpha,0}$ are given in Definition 2.4.

Proof. Since A generates the resolvent family $\{S^a(t)\}_{t\geq 0}$, by Theorem 3.12 we conclude that the solution to (4.26) is given

$$\begin{aligned}u^n &= S^{\kappa,n}_{\alpha,1}x+\tau(S^\kappa_{\alpha,\alpha}\star f)^n\\ &= \int_0^\infty\int_0^\infty\rho_n^\tau(t)\psi_{\alpha,1-\alpha}(t,s)S^a(s)xdsdt+\tau\sum_{j=0}^{n}\int_0^\infty\int_0^\infty\rho_{n-j}^\tau(t)\psi_{\alpha,0}(t,s)S^a(s)dsdtf^j,\end{aligned}$$

which can be written as (4.29) by (4.28). □

Now, let $-A$ be a non-negative and self-adjoint operator on the Hilbert space $L^2(\Omega)$, where $\Omega\subset\mathbb{R}^N$ is a bounded open set. Assume that A has a compact resolvent. Then $-A$ has a discrete spectrum and the corresponding eigenvalues satisfy $0\leq\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq\cdots$ with $\lim_{n\to\infty}\lambda_n=\infty$.

If ϕ_n denotes the normalized eigenfunction associated with λ_n, then $\{\phi_n : n\in\mathbb{N}\}$ is an orthonormal basis for $L^2(\Omega)$, and for all $v\in D(A)$ we can write

$$-Av=\sum_{k=1}^{\infty}\lambda_n\langle v,\phi_n\rangle_{L^2(\Omega)}\phi_n.$$

Proposition 4.18. *Let A be an operator as above. Suppose that $\kappa(t)=\gamma\frac{t^{\mu-1}}{\Gamma(\mu)}e^{-\rho t}$, where $\gamma\in\mathbb{R}\setminus\{0\}$. Assume that $f(t,\cdot)\in L^2(\Omega)$ for all $t\geq 0$. If $0<\mu<\frac{1}{2}$ and $\frac{1}{2}<\alpha<1$, then the unique solution u to the semidiscrete Problem*

$$_C\nabla^\alpha u^n(x)=Au^n(x)+\sum_{j=0}^{n}\kappa^{n-j}Au^j(x)+f^n(x)\tag{4.30}$$

where $x\in\Omega$, under the initial condition $u^0=u_0(x)$ and $u_0\in L^2(\Omega)$ is given by

$$\begin{aligned}u^n=&\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}\frac{(-\lambda_m)^i\gamma^i}{i!}\int_0^\infty\int_0^\infty\rho_n^\tau(t)\psi_{\alpha,1-\alpha}(t,s)\times\\ &\int_0^s(s-r)^ie^{-\lambda_m(s-r)}r^{\frac{\mu i}{\alpha}-1}E^{\mu i}_{\frac{1}{\alpha},\frac{\mu i}{\alpha}}(-\rho r^{\frac{1}{\alpha}})u_{0,m}\phi_m(x)drdsdt\\ &+\tau\sum_{m=1}^{\infty}\sum_{j=0}^{n}\sum_{i=0}^{\infty}\frac{(-\lambda_m)^i\gamma^i}{i!}\int_0^\infty\int_0^\infty\rho_n^\tau(t)\psi_{\alpha,0}(t,s)\times\\ &\int_0^s(s-r)^ie^{-\lambda_m(s-r)}r^{\frac{\mu i}{\alpha}-1}E^{\mu i}_{\frac{1}{\alpha},\frac{\mu i}{\alpha}}(-\rho r^{\frac{1}{\alpha}})f^j_m\phi_m(x)drdsdt,\end{aligned}\tag{4.31}$$

for all $n\in\mathbb{N}$, where for $m\in\mathbb{N}$, $u_{0,m}:=\langle u_0(\cdot),\phi_m(\cdot)\rangle_{L^2(\Omega)}$, and $f_m(t):=\langle f(t,\cdot),\phi_m(\cdot)\rangle_{L^2(\Omega)}$.

Proof. Consider the problem

$$\begin{cases}\partial_t^\alpha u(t,x)=Au(t,x)+\int_0^t\kappa(t-s)Au(s,x)ds+f(t,x),\ t\geq 0,x\in\Omega\\ u(0,x)=u_0(x),\quad x\in\Omega.\end{cases}\tag{4.32}$$

Multiplying both sides of (4.32) by $\phi_m(x)$ and integrating over Ω we obtain

$$\begin{cases} \partial_t^\alpha u_m(t) = -\lambda_m u_m(t) - \lambda_m \int_0^t \kappa(t-s)u_m(s)ds + f_m(t), \;\; t>0, \\ u_m(0) = u_{0,m}, \end{cases} \tag{4.33}$$

for all $m \in \mathbb{N}$, where the functions u_m, ϕ_m and f_m are defined by $u_m(t) := \langle u(t,\cdot), \phi_m(\cdot)\rangle_{L^2(\Omega)}$, $u_{0,m} := \langle u_0(\cdot), \phi_m(\cdot)\rangle_{L^2(\Omega)}$, and $f_m(t) := \langle f(t,\cdot), \phi_m(\cdot)\rangle_{L^2(\Omega)}$. Multiplying (4.33) by $\rho_n^\tau(t)$ and integrating over $[0,\infty)$ we get

$$_C\nabla^\alpha u_m^n = -\lambda_j u_m^n - \lambda_m \gamma \sum_{l=0}^{n} k_\tau^\mu(n-l)\frac{1}{(1+\rho\tau)^{n-l+\mu}} u_m^l + f_m^n, \quad n \in \mathbb{N}. \tag{4.34}$$

By Proposition 4.17, the solution u_m^n to (4.34) under the initial condition $u^0 = u_{0,m}$ is given by (4.29), where ϱ is replaced by $-\lambda_m$, x by $u_{0,m}$ and f by f_m.

On the other hand, an easy computation shows that κ is a 1-regular kernel of a constant μ and $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\mathrm{Re}(\lambda) > -\rho$, where a is defined in (4.27). Since $0 < \mu < \frac{1}{2}$ and $A \in \mathrm{Sec}(\theta, 1)$ for all $\theta \in (\pi/2, \pi)$, we obtain $(1+\mu/\alpha)\frac{\pi}{2} \leq \theta$. Since $u(t,x) = \sum_{m=1}^\infty u_m(t)\phi_m(x)$, we obtain for all $x \in \Omega$ that

$$u^n(x) = \int_0^\infty \rho_n^\tau(t)u(t,x)dt = \sum_{m=1}^\infty \int_0^\infty \rho_n^\tau(t)u_m(t)dt\phi_m(x) = \sum_{m=1}^\infty u_m^n \phi_m(x),$$

which can be written as (4.31).

□

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