EXISTENCE OF MILD SOLUTIONS TO NONLOCAL FRACTIONAL CAUCHY PROBLEMS VIA COMPACTNESS.

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Abstract. In this paper we obtain characterizations of compactness for resolvent families of operators and as applications we study the existence of mild solutions to nonlocal Cauchy problems for fractional derivatives in Banach spaces. We discuss here simultaneously the Caputo and Riemann-Liouville fractional derivatives in the cases $0 < \alpha < 1$ and $1 < \alpha < 2$.

1. Introduction

The nonlocal initial conditions were introduced to extend the classical theory of initial value problems. Nonlocal conditions describe more appropriately some natural phenomena because they consider additional information in the initial conditions.

The existence of mild solutions to semilinear Cauchy problems with nonlocal conditions has been studied by several authors in the last two decades. See for instance [3, 5, 7, 10] and the references cited therein.

On the other hand, many authors have studied recently the existence of mild solutions to abstract fractional differential equations with nonlocal conditions by using the theory of resolvent families of operators as well as some fixed point results. See [2, 4, 6, 11, 13, 20, 21, 24, 27, 28, 29, 31, 32, 33] and the references therein for more details.

Let A be a closed and linear operator defined on a Banach space X, $u_0, u_1 \in X$, $T > 0$ and suppose that f, p, q are suitable continuous functions. In what follows, we will denote by D_t^α and D^α, to the Caputo and Riemann-Liouville fractional derivatives, respectively. Now, for $t \in [0, T]$ we consider the following nonlinear fractional differential equations with nonlocal conditions

$$D_t^\alpha u(t) = Au(t) + f(t, u(t)), \quad u(0) = p(u) + u_0, \tag{1.1}$$

and

$$D^\alpha u(t) = Au(t) + f(t, u(t)),\ (g_{1-\alpha} * u)(0) = p(u) + u_0, \tag{1.2}$$

in case $0 < \alpha < 1$; and

$$D_t^\alpha u(t) = Au(t) + f(t, u(t)), \quad u(0) = p(u) + u_0, \quad u'(0) = q(u) + u_1, \tag{1.3}$$

and

$$D^\alpha u(t) = Au(t) + f(t, u(t)),\ (g_{2-\alpha} * u)(0) = p(u) + u_0,\ (g_{2-\alpha} * u)'(0) = q(u) + u_1, \tag{1.4}$$

in case $1 < \alpha < 2$.

By using the Laplace transform it is easy to see that the *mild* solution to problems (1.1)–(1.4) are respectively given by

$$u(t) \quad = \quad S_{\alpha,1}(t)(u_0 + p(u)) + \int_0^t S_{\alpha,\alpha}(t-s) f(s, u(s))ds, \tag{1.5}$$

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$$u(t)=S_{\alpha,\alpha}(t)(u_0+(u))+\int_0^t S_{\alpha,\alpha}(t-s)f(s,u(s))ds, \tag{1.6}$$

in case $0<\alpha<1$; and

$$u(t) \;=\; S_{\alpha,1}(t)(u_0+p(u))+S_{\alpha,2}(t)(u_1+q(u))+\int_0^t S_{\alpha,\alpha}(t-s)f(s,u(s))ds, \tag{1.7}$$

and

$$u(t)=S_{\alpha,\alpha-1}(t)(u_0+p(u))+S_{\alpha,\alpha}(t)(u_1+q(u))+\int_0^t S_{\alpha,\alpha}(t-s)f(s,u(s))ds, \tag{1.8}$$

in case $1<\alpha<2$. Here, for $\alpha,\beta>0$, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is the *resolvent family* generated by A (see definition below, Section 2).

The existence of mild solutions to problems (1.1)-(1.4) has been studied by many authors in the last years. For example, in case $0<\alpha<1$ we refer to the reader to [11, 13, 32, 33] (for the Caputo fractional derivative) and to [20] (for the Riemann-Liouville fractional derivative), that is, the problems (1.1) and (1.2), respectively. On the other hand, in case $1<\alpha<2$, the existence of mild solutions to the Caputo fractional Cauchy problems with nonlocal conditions (1.3) has been considered in [15, 24], and the references therein, and to the best of our knowledge, the nonlocal Riemann-Liouville fractional Cauchy problem (1.4) has not been addressed in the existing literature.

A common assumption in many of the above mentioned papers to obtain the existence of mild solutions to problems (1.1)–(1.4) is that A generates a compact analytic semigroup $\{T(t)\}_{t\geq 0}$, or A generates a compact *fractional* resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$, (see the definition below) because the compactness of $\{T(t)\}_{t\geq 0}$ (or $\{S_{\alpha,1}(t)\}_{t\geq 0}$) allows to apply, for example, the Krasnoselskii fixed point theorem.

According to the variation of constants formulas (1.5)–(1.8) we observe that if we have a compactness criteria of $S_{\alpha,\beta}(t)$ (for suitable α and β) we will be able to apply some fixed point techniques to obtain the existence of mild solutions to problems (1.1)–(1.4). For example, to prove the existence of mild solutions to problem (1.3) the authors in [24, Theorem 1.2] assume that the operators $S_{\alpha,1}(t), S_{\alpha,2}(t)$ and $S_{\alpha,\alpha}(t)$ generated by A are compact for all $t>0$. However, there are not completely clear conditions on A implying the compactness of $S_{\alpha,1}(t), S_{\alpha,2}(t)$ and $S_{\alpha,\alpha}(t)$ for all $t>0$, because there is not a compactness criteria for $S_{\alpha,\beta}(t)$, when $\alpha,\beta>0$. Therefore, we notice that the compactness of $S_{\alpha,\beta}(t)$ gives a powerful tool to obtain existence of mild solutions to problems (1.1)–(1.4).

The compactness of $S_{\alpha,\beta}(t)$ it is well known in some special cases. For example, if $\alpha=\beta=1$, then $S_{1,1}(t)$ is compact for all $t>0$ if and only if $S_{1,1}(t)$ is norm continuos and $(\lambda-A)^{-1}$ is compact for all $\lambda\in\rho(A)$, because $\{S_{1,1}(t)\}_{t\geq 0}$ corresponds to a C_0-semigroup. See [22, Theorem 3.3, Chapter 2]. If $\alpha=\beta=2$, then $S_{2,2}(t)$ is compact for all $t>0$ if and only if $(\lambda^2-A)^{-1}$ is compact $\lambda\in\rho(A)$, because $\{S_{2,2}(t)\}_{t\geq 0}$ is the sine family generated by A, see [25]. In case $0<\alpha<1$, the compactness of $S_{\alpha,1}(t)$ has been studied by using subordination methods, that is, the operator A is supposed to be a generator of a compact semigroup, see [23]. On the other hand, if A is an almost sectorial operator and the resolvent $(\lambda^\alpha-A)^{-1}$ is a compact for all $\lambda\in\rho(A)$, then $S_{\alpha,1}(t)$ is compact for all $t>0$, (see [26]), and very recently, was proved that if $S_{\alpha,1}(t)$ is norm continuous, then $S_{\alpha,1}(t)$ compact for all $t>0$ if and only if $(\lambda^\alpha-A)^{-1}$ is compact for all $\lambda\in\rho(A)$. See [9, 18]. Finally, in case $1<\alpha<2$ the characterization of compactness asserts that $S_{\alpha,\alpha}(t)$ is compact for all $t>0$ if and only if $(\lambda^\alpha-A)^{-1}$ is compact for all $\lambda\in\rho(A)$, see [18, Theorem 3.5].

In this paper, we study the existence of mild solution to the nonlocal fractional Cauchy problems (1.1)–(1.4). Our approach relies on the compactness of resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ for suitable $\alpha,\beta>0$, as well as some fixed point techniques. We remark that we study simultaneously the nonlocal fractional Cauchy problem for the Caputo and Riemann-Liouville fractional derivatives.

The paper is organized as follows. The Section 2 gives the Preliminaries. Section 3 is devoted to the norm continuity and compactness of $S_{\alpha,\beta}(t)$ for $t>0$. Here, we give characterizations of the compactness of $S_{\alpha,\beta}(t)$ for $t>$ for suitable $\alpha,\beta>0$. In Section 4 we study nonlocal fractional Cauchy problems for the Caputo fractional derivative. We give some results on the existence of mild solutions to Problems (1.1) and (1.3). The Section 5 treats nonlocal fractional Cauchy problems for the Riemann-Liouville fractional derivative. Here, we study the existence of mild solutions to Problems (1.2) and (1.4). Finally, the Section 6 is devoted to some applications.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. We denote by $\mathcal{B}(X)$ the space of all bounded linear operators from X into X. If A is a closed linear operator on X we denote by $\rho(A)$ the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent operator of A defined for all $\lambda \in \rho(A)$.

We recall that a strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be of type (M, ω) or *is exponentially bounded,* if there exist two constants $M > 0$ and $w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{wt}$ for all $t > 0$.

Now, we review some results on fractional calculus. For $\mu > 0$, define

$$g_\mu(t) = \begin{cases} \frac{t^{\mu-1}}{\Gamma(\mu)}, & t > 0 \\ 0, & t \leq 0, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function. We define $g_0 \equiv \delta_0$, the Dirac delta. For $\mu > 0$, $n = \lceil \mu \rceil$ denotes the smallest integer n greater than or equal to μ. As usual, the finite convolution of f and g is defined by $(f * g)(t) = \int_0^t f(t-s)g(s)ds$.

Definition 2.1. *Let $\alpha > 0$. The α-order Riemann-Liouville fractional integral of u is defined by*

$$J^\alpha u(t) := \int_0^t g_\alpha(t-s)u(s)ds, \quad t \geq 0.$$

Also, we define $J^0 u(t) = u(t)$. Because of the convolution properties, the integral operators $\{J^\alpha\}_{\alpha\geq 0}$ satisfy the semigroup law: $J^\alpha J^\beta = J^{\alpha+\beta}$ for all $\alpha, \beta \geq 0$.

Definition 2.2. *Let $\alpha > 0$. The α-order Caputo fractional derivative is defined*

$$D_t^\alpha u(t) := \int_0^t g_{n-\alpha}(t-s)u^{(n)}(s)ds,$$

where $n = \lceil \alpha \rceil$.

Definition 2.3. *Let $\alpha > 0$. The α-order Riemann-Liouville fractional derivative of u is defined*

$$D^\alpha u(t) := \frac{d^n}{dt^n}\int_0^t g_{n-\alpha}(t-s)u(s)ds,$$

where $n = \lceil \alpha \rceil$.

We notice that if $\alpha = m \in \mathbb{N}$, then $D_t^m = D^m = \frac{d^m}{dt^m}$.

Throughout this paper we use the notation D_t^α and D^α to the α-fractional derivative of Caputo and Riemann-Liouville, respectively.

Example 2.4. *If $\alpha, \beta > 0$, then*

i) $J^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$,

ii) $D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\beta-\alpha} = D_t^\alpha t^\beta$.

iii) $D_t^\alpha e^{\rho t} = \rho^2 t^{2-\alpha} e_{1,3-\alpha}(\rho t)$.

We observe that the Riemann-Liouville derivative operator D^α is a left inverse operator of J^α but not a right inverse, that is,

$$D^\alpha J^\alpha u(t) = u(t),$$

and

$$(J^\alpha D^\alpha)u(t) = u(t) - \sum_{k=0}^{n-1}(g_{n-\alpha} * u)^{(k)}(0)g_{\alpha+1+k-n}(t),$$

$n = \lceil\alpha\rceil$. On the other hand, the Caputo derivative operator D_t^α satisfies

$$D_t^\alpha J^\alpha u(t) = u(t),$$

and

$$(J^\alpha D_t^\alpha)u(t) = u(t) - \sum_{k=0}^{n-1} u^{(k)}(0)g_{k+1}(t).$$

If we denote by $\hat{f}$ (or $\mathcal{L}(f)$) to the Laplace transform of f, we have the following properties for the fractional derivatives

$$\widehat{D^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{n-1}(g_{n-\alpha} * u)^{(k)}(0)\lambda^{n-1-k} \tag{2.1}$$

and

$$\widehat{D_t^\alpha u}(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{n-1} u^{(k)}(0)\lambda^{\alpha-1-k}, \tag{2.2}$$

where $n = \lceil\alpha\rceil$ and $\lambda \in \mathbb{C}$. For $\alpha, \beta > 0$ and $z \in \mathbb{C}$, the generalized Mittag-Leffler function is defined by

$$e_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

The Laplace transform $\mathcal{L}$ of the Mittag-Leffler function satisfies

$$\mathcal{L}(t^{\beta-1}e_{\alpha,\beta}(\rho t^\alpha))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \rho}, \quad \rho \in \mathbb{C}, \mathrm{Re}\lambda > |\rho|^{1/\alpha}.$$

Definition 2.5. *Let A be closed linear operator with domain $D(A)$, defined on a Banach space X, and $\alpha, \beta > 0$. We say that A is the generator of an (α,β)-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha,\beta} : [0,\infty) \to \mathcal{B}(X)$ such that $\{S_{\alpha,\beta}(t)\}$ is exponentially bounded, $\{\lambda^\alpha : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$, and for all $x \in X$,*

$$\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t}S_{\alpha,\beta}(t)x\,dt, \quad \mathrm{Re}\lambda > \omega.$$

In this case, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is called the (α,β)-resolvent family *generated by A.*

We notice that the Definition 2.5 corresponds to the concept of (a,k)-regularized families introduced in [16]. In fact, if $a = g_\alpha$ and $b = g_\beta$ then the function $t \mapsto S_{\alpha,\beta}(t)$, is a (g_α, g_β)-regularized family. Moreover, the function $S_{\alpha,\beta}(t)$ satisfies the following functional equation (see [14, 19]):

$$S_{\alpha,\beta}(s)(g_\alpha * S_{\alpha,\beta})(t) - (g_\alpha * S_{\alpha,\beta})(s)S_{\alpha,\beta}(s) = g_\beta(s)(g_\alpha * S_{\alpha,\beta})(t) - g_\beta(t)(g_\alpha * S_{\alpha,\beta})(s),$$

for all $t, s \geq 0$. On the other hand, if an operator A with domain $D(A)$ is the infinitesimal generator of the (α,β)-resolvent family $S_{\alpha,\beta}(t)$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t\to 0^+} \frac{S_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)}.$$

For example, the case $S_{1,1}(t)$ corresponds to a C_0-semigroup, $S_{2,1}(t)$ is a cosine family, whereas $S_{2,2}(t)$ is a sine family. Finally, if $\beta = 1$, then $S_{\alpha,1}(t)$ is the α-resolvent family (also called the α-times resolvent family) for fractional differential equations. We notice that in the scalar case, that is, when $A = \rho I$, where $\rho \in \mathbb{C}$ and I denotes the identity operator, then by the uniqueness of the Laplace transform

$S_{\alpha,\beta}(t)$ corresponds to the function $t^{\beta-1}e_{\alpha,\beta}(\rho t^{\alpha})$. Finally, let $0<\alpha<1$ and $\beta\geq\alpha$. Define $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ by

$$S_{\alpha,\beta}(t)f(s):=\int_0^s f(s-r)\varphi_{\alpha,\beta-\alpha}(t,r)dr,$$

where $s\in\mathbb{R}_+$, $f\in L^1(\mathbb{R}_+)$ and the function $\varphi_{a,b}(t,r)$ is defined by

$$\varphi_{a,b}(t,r):=t^{b-1}W_{-a,b}(-rt^{-a}),\quad a>0,b\geq 0, \tag{2.3}$$

where $W_{-a,b}(z):=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(-an+b)}$ $(z\in\mathbb{C})$ denotes the Wright function. Then, $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is an (α,β)-resolvent family on the Banach space $X=L^1(\mathbb{R}_+)$ generated by $A=-\frac{d}{dt}$. See [1, Example 11].

The proof of the next result follows as in [14, 16].

Proposition 2.6. *Let $\alpha,\beta>0$ and let $\{S_{\alpha,\beta}(t)\}_{t\geq 0}\subset\mathcal{B}(X)$ be an (α,β)-resolvent family generated by A. Then the following holds:*

1. *$S_{\alpha,\beta}(t)x\in D(A)$ and $S_{\alpha,\beta}(t)Ax=AS_{\alpha,\beta}(t)x$ for all $x\in D(A)$ and $t\geq 0$.*
2. *If $x\in D(A)$ and $t\geq 0$, then*

$$S_{\alpha,\beta}(t)x=g_\beta(t)x+\int_0^t g_\alpha(t-s)AS_{\alpha,\beta}(s)xds. \tag{2.4}$$

3. *If $x\in X$ and $t\geq 0$, then $\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds\in D(A)$, and*

$$S_{\alpha,\beta}(t)x=g_\beta(t)x+A\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds.$$

In particular, $S_{\alpha,\beta}(0)=g_\beta(0)I$.

Finally, we recall the following results.

Theorem 2.7 (Mazur Theorem)**.** *If K is a compact subset of a Banach space X, then its convex closure $\overline{\mathrm{conv}(K)}$ is compact.*

Theorem 2.8 (Krasnoselskii Theorem)**.** *Let C be a closed convex and nonempty subset of a Banach space X. Let Q_1 and Q_2 be two operators such that*

i) *If $u,v\in C$, then $Q_1u+Q_2v\in C$.*
ii) *Q_1 is a mapping contraction.*
iii) *Q_2 is compact and continuous.*

Then, there exists $z\in C$ such that $z=Q_1z+Q_2z$.

Theorem 2.9 (Schauder's fixed point Theorem)**.** *Let C be a nonempty, closed, bounded and convex subset of a Banach space X. Suppose that $\Gamma:C\to C$ is a compact operator. Then Γ has at least a fixed point in C.*

Theorem 2.10 (Leray-Schauder Alternative Theorem)**.** *Let C be a convex subset of a Banach space X. Suppose that $0\in C$. If $\Gamma:C\to C$ is a completely continuous map, then either Γ has a fixed point, or the set $\{x\in C: x=\lambda\Gamma(x), 0<\lambda<1\}$ is unbounded.*

3. Continuity and Compactness of $S_{\alpha,\beta}(t)$.

In this section we study, for all $t>0$, the norm continuity (continuity in $\mathcal{B}(X)$) and the compactness of $S_{\alpha,\beta}(t)$ for given $\alpha,\beta>0$.

Proposition 3.11. *Let $\alpha>0$ and $1<\beta\leq 2$. Suppose that $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is the (α,β)-resolvent family of type (M,ω) generated by A. Then the function $t\mapsto S_{\alpha,\beta}(t)$ is continuous in $\mathcal{B}(X)$ for all $t>0$.*

Proof. Let $1<\beta<2$. Observe that for all $\mathrm{Re}\lambda>0$,

$$\mathcal{L}(S_{\alpha,\beta})(\lambda) = \lambda^{\alpha-\beta}(\lambda^\alpha-A)^{-1}=\frac{1}{\lambda^{\beta-1}}\lambda^{\alpha-1}(\lambda^\alpha-A)^{-1}=\mathcal{L}(g_{\beta-1}*S_{\alpha,1})(\lambda).$$

We conclude by the uniqueness of the Laplace transform that $S_{\alpha,\beta}(t)=(g_{\beta-1}*S_{\alpha,1})(t)$, for all $t>0$. Take $0<t_0<t_1$. Then

$$\begin{aligned} S_{\alpha,\beta}(t_1)-S_{\alpha,\beta}(t_0) &= (g_{\beta-1}*S_{\alpha,1})(t_1)-(g_{\beta-1}*S_{\alpha,1})(t_0)\\ &= \int_{t_0}^{t_1} g_{\beta-1}(t_1-r)S_{\alpha,1}(r)dr+\int_0^{t_0}[g_{\beta-1}(t_1-r)-g_{\beta-1}(t_0-r)]S_{\alpha,1}(r)dr\\ &=: I_1+I_2. \end{aligned}$$

Since $\beta>1$, we have $g_\beta(0)=0$ and we obtain

$$\|I_1\| \le \int_{t_0}^{t_1} g_{\beta-1}(t_1-r)\|S_{\alpha,1}(r)\|dr\le M\int_{t_0}^{t_1} g_{\beta-1}(t_1-r)e^{\omega r}dr=Me^{\omega t_1}g_\beta(t_1-t_0),$$

and therefore $\|I_1\|\to 0$ as $t_1\to t_0$.

On the other hand,

$$\begin{aligned} \|I_2\| &\le \int_0^{t_0} |g_{\beta-1}(t_1-r)-g_{\beta-1}(t_0-r)|\|S_{\alpha,1}(r)\|dr\\ &\le Me^{\omega t_1}\int_0^{t_0} |g_{\beta-1}(t_1-r)-g_{\beta-1}(t_0-r)|dr\\ &= Me^{\omega t_1}\int_0^{t_0} |g_{\beta-1}(t_1-t_0+r)-g_{\beta-1}(r)|dr. \end{aligned}$$

Since $1<\beta<2$ we obtain that the function $r\mapsto g_{\beta-1}(r)$ is decreasing in $[0,\infty)$ and therefore $g_{\beta-1}(r)-g_{\beta-1}(t_1-t_0+r)>0$, for all $r>0$, obtaining

$$\|I_2\| \le Me^{\omega t_1}\int_0^{t_0}[g_{\beta-1}(r)-g_{\beta-1}(t_1-t_0+r)]dr=Me^{\omega t_1}[g_\beta(t_0)-g_\beta(t_1)+g_\beta(t_1-t_0)].$$

Therefore, $\|I_2\|\to 0$ as $t_1\to t_0$. We conclude that $S_{\alpha,\beta}(t)$ is norm continuous, for $1<\beta<2$.

On the other hand, if $\beta=2$, then by the uniqueness of the Laplace transform, we obtain that

$$S_{\alpha,2}(t)x=(g_1*S_{\alpha,1})(t)x=\int_0^t S_{\alpha,1}(r)xdr,$$

for all $x\in X$. Take $0<t_0<t_1$. Then

$$\|S_{\alpha,2}(t_1)x-S_{\alpha,2}(t_0)x\|\le\int_{t_0}^{t_1}\|S_{\alpha,1}(r)x\|dr\le Me^{\omega t_1}\|x\|(t_1-t_0),$$

for all $x\in X$. Therefore $\|S_{\alpha,2}(t_1)-S_{\alpha,2}(t_0)\|\to 0$ as $t_1\to t_0$. □

Lemma 3.12. *Suppose that A generates an (α,β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t\ge 0}$ of type (M,ω). If $\gamma>0$, then A generates an $(\alpha,\beta+\gamma)$-resolvent family of type $\left(\frac{M}{\omega^\gamma},\omega\right)$.*

Proof. By hypothesis we get for all $t\ge 0$,

$$\begin{aligned} \|(g_\gamma*S_{\alpha,\beta})(t)\| &\le M\int_0^t g_\gamma(t-s)e^{\omega s}ds\le Me^{\omega t}\int_0^t g_\gamma(s)e^{-\omega s}ds\\ &\le Me^{\omega t}\int_0^\infty g_\gamma(s)e^{-\omega s}ds=\frac{Me^{\omega t}}{\omega^\gamma}. \end{aligned}$$

Therefore $(g_\gamma * S_{\alpha,\beta})(t)$ is Laplace transformable and for all $\lambda > \omega$, we have

$$\mathcal{L}(g_\gamma * S_{\alpha,\beta})(\lambda) = \frac{1}{\lambda^\gamma}\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1} = \lambda^{\alpha-(\beta+\gamma)}(\lambda^\alpha - A)^{-1} = \mathcal{L}(S_{\alpha,\beta+\gamma})(\lambda).$$

We conclude that A generates an $(\alpha, \beta+\gamma)$-resolvent family of type $\left(\frac{M}{\omega^\gamma}, \omega\right)$. □

Definition 3.13. *We say that the resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is* compact *if for every $t > 0$, the operator $S_{\alpha,\beta}(t)$ is a compact operator.*

In what follows, we will assume that $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is strongly continuous for all $\alpha, \beta > 0$.

Theorem 3.14. *Let $\alpha > 0$, $1 < \beta \leq 2$ and $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ be an (α,β)-resolvent family of type (M,ω) generated by A. Then the following assertions are equivalent*

i) *$S_{\alpha,\beta}(t)$ is a compact operator for all $t > 0$.*
ii) *$(\mu - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that the resolvent family $\{S_{\alpha,\beta}(t)\}_{t>0}$ is compact. Let $\lambda > \omega$ be fixed. Then we have

$$\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1} = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)dt,$$

where the integral in the right-hand side exists in the Bochner sense, because $\{S_{\alpha,\beta}(t)\}_{t>0}$ is continuous in the uniform operator topology (by Proposition 3.11) we conclude that $(\lambda^\alpha - A)^{-1}$ is a compact operator by [30, Corollary 2.3].

$(ii) \Rightarrow (i)$ Let $t > 0$ be fixed. Assume that $1 < \beta < 2$. Since $\beta > 1$, it follows that $g_{\beta-1} \in L^1_{\text{loc}}[0,\infty)$ and therefore, by [12, Proposition 2.1] we obtain

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{\omega-iN}^{\omega+iN} e^{\lambda t}(\widehat{g_{\beta-1} * S_{\alpha,1}})(\lambda)d\lambda = (g_{\beta-1} * S_{\alpha,1})(t) = S_{\alpha,\beta}(t),$$

in $\mathcal{B}(X)$. Therefore,

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}d\lambda = S_{\alpha,\beta}(t), \quad t > 0,$$

where Γ is the path consisting of the vertical line $\{\omega + is : s \in \mathbb{R}\}$. By hypothesis and [30, Corollary 2.3], we conclude that $S_{\alpha,\beta}(t)$ is compact for all $\alpha > 0$ and $1 < \beta < 2$. Now, we take $\beta = 2$. Observe that in $\mathcal{B}(X)$ we have

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{\omega-iN}^{\omega+iN} e^{\lambda t}(\widehat{g_1 * S_{\alpha,1}})(\lambda)d\lambda = (g_1 * S_{\alpha,1})(t) = S_{\alpha,2}(t),$$

by [12, Proposition 2.1], and as in case $1 < \beta < 2$ we conclude that $S_{\alpha,2}(t)$ is compact for all $t > 0$. □

By Theorem 3.14 we have the following Corollary.

Corollary 3.15. *Let $1 < \alpha \leq 2$ and $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ be an (α,α)-resolvent family of type (M,ω) generated by A. Then the following assertions are equivalent*

i) *$S_{\alpha,\alpha}(t)$ is a compact operator for all $t > 0$.*
ii) *$(\mu - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.*

Proposition 3.16. *Let $1 < \alpha < 2$, and $\{S_{\alpha,1}(t)\}_{t\geq 0}$ be the $(\alpha,1)$-resolvent family of type (M,ω) generated by A. Suppose that $S_{\alpha,1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Then the following assertions are equivalent*

i) *$S_{\alpha,1}(t)$ is a compact operator for all $t > 0$.*
ii) *$(\mu - A)^{-1}$ is a compact operator for all $\mu > \omega^{1/\alpha}$.*

Proof. $(i) \Rightarrow (ii)$ Suppose that that the resolvent family $\{S_{\alpha,1}(t)\}_{t>0}$ is compact. Let $\lambda > \omega$ be fixed. Then we have

$$\lambda^{\alpha-1}(\lambda^{\alpha} - A)^{-1} = \int_0^{\infty} e^{-\lambda t} S_{\alpha,1}(t)dt,$$

where the integral in the right-hand side exists in the Bochner sense, because $\{S_{\alpha,1}(t)\}_{t>0}$ is continuous in the uniform operator topology, by hypothesis. Then, by [30, Corollary 2.3] we conclude that $(\lambda^{\alpha} - A)^{-1}$ is a compact operator.

$(ii) \Rightarrow (i)$ Let $t > 0$ be fixed. Since $1 < \alpha < 2$, it follows that $g_{2-\alpha} \in L^1_{\text{loc}}[0, \infty)$ and therefore, by [12, Proposition 2.1] we obtain

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega+iN} e^{\lambda t} (\widehat{g_{2-\alpha} * S_{\alpha,\alpha-1}})(\lambda)d\lambda = (g_{2-\alpha} * S_{\alpha,\alpha-1})(t) = S_{\alpha,1}(t),$$

in $\mathcal{B}(X)$. Therefore,

$$\frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1}(\lambda^{\alpha} - A)^{-1} d\lambda = S_{\alpha,1}(t), \quad t > 0,$$

where Γ is the path consisting of the vertical line $\{\omega + is : s \in \mathbb{R}\}$. By hypothesis and [30, Corollary 2.3], we conclude that $S_{\alpha,1}(t)$ is compact. □

Proposition 3.17. *Let* $\frac{3}{2} < \alpha < 2$, *and* $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ *be the* $(\alpha, \alpha - 1)$*-resolvent family of type* (M, ω) *generated by* A. *Suppose that* $S_{\alpha,\alpha-1}(t)$ *is continuous in the uniform operator topology for all* $t > 0$. *Then the following assertions are equivalent*

i) $S_{\alpha,\alpha-1}(t)$ *is a compact operator for all* $t > 0$.
ii) $(\mu - A)^{-1}$ *is a compact operator for all* $\mu > \omega^{1/\alpha}$.

Proof. $(i) \Rightarrow (ii)$ It follows as in the proof of Proposition 3.16

$(ii) \Rightarrow (i)$ Let $t > 0$ be fixed. Since $\alpha > 3/2$, it follows that $g_{\alpha-\frac{3}{2}} \in L^1_{\text{loc}}[0, \infty)$ and therefore, by [12, Proposition 2.1] we obtain

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega+iN} e^{\lambda t} (\widehat{g_{\alpha-\frac{3}{2}} * S_{\alpha,\frac{1}{2}}})(\lambda)d\lambda = (g_{\alpha-\frac{3}{2}} * S_{\alpha,\frac{1}{2}})(t) = S_{\alpha,\alpha-1}(t),$$

in $\mathcal{B}(X)$. Therefore,

$$\frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1}(\lambda^{\alpha} - A)^{-1} d\lambda = S_{\alpha,\alpha-1}(t),$$

where Γ is the path consisting of the vertical line $\{\omega + is : s \in \mathbb{R}\}$. By hypothesis and [30, Corollary 2.3], we conclude that $S_{\alpha,\alpha-1}(t)$ is compact. □

The proof of the next result follows similarly to Proposition 3.16, because for $1/2 < \alpha < 1$ we have

$$\lim_{N\to\infty} \frac{1}{2\pi i} \int_{\omega-iN}^{\omega+iN} e^{\lambda t} (\widehat{g_{\alpha-\frac{1}{2}} * S_{\alpha,\frac{1}{2}}})(\lambda)d\lambda = (g_{\alpha-\frac{1}{2}} * S_{\alpha,\frac{1}{2}})(t) = S_{\alpha,\alpha}(t),$$

in $\mathcal{B}(X)$ and $t > 0$ by [12, Proposition 2.1].

Proposition 3.18. *Let* $1/2 < \alpha < 1$, *and* $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ *be the* (α, α)*-resolvent family of type* (M, ω) *generated by* A. *Suppose that* $S_{\alpha,\alpha}(t)$ *is continuous in the uniform operator topology for all* $t > 0$. *Then, the following assertions are equivalent*

i) $S_{\alpha,\alpha}(t)$ *is a compact operator for all* $t > 0$.
ii) $(\mu - A)^{-1}$ *is a compact operator for all* $\mu > \omega^{1/\alpha}$.

Remark 3.19. *Let $\varepsilon_0 > 0$ be fixed. If $\varepsilon_0 < \alpha < 1$, then by* [12, Proposition 2.1] *we have*

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{\omega-iN}^{\omega+iN} e^{\lambda t}(\widehat{g_{\alpha-\varepsilon_0} * S_{\alpha,\varepsilon_0}})(\lambda)d\lambda = (g_{\alpha-\varepsilon_0} * S_{\alpha,\varepsilon_0})(t) = S_{\alpha,\alpha}(t),$$

in $\mathcal{B}(X)$. Therefore, as is Proposition 3.18, if $\alpha > \varepsilon_0$, where $\varepsilon_0 > 0$, A generates the (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ of type (M,ω) and $S_{\alpha,\alpha}(t)$ is norm continuous for all $t > 0$, then $S_{\alpha,\alpha}(t)$ is a compact operator for all $t > 0$ if and only if $(\lambda^\alpha - A)^{-1}$ is a compact operator for all $\lambda > \omega^{1/\alpha}$. The same conclusion holds if $\varepsilon_0 < \alpha < 2$, where $\varepsilon_0 > 1$ is fixed and $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ is the $(\alpha,\alpha-1)$-resolvent family of type (M,ω) generated by A, which is norm continuous for all $t > 0$.

4. Non-local fractional Cauchy problems. The Caputo case.

In this section we consider the non-local problem for the Caputo fractional derivative

$$\left\{\begin{array}{lcl} D_t^\alpha u(t) & = & Au(t) + f(t,u(t)), \quad t \in I := [0,T] \\ u(0)+p(u) & = & u_0 \\ u'(0)+q(u) & = & u_1, \end{array}\right. \tag{4.5}$$

$u_0, u_1 \in X$, $1 < \alpha < 2$, $T > 0$ and A is a closed linear operator defined on X which generates the $(\alpha,1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$. The nonlinear function $f : [0,T]\times X \to X$ is continuous and the nonlocal conditions $p, q : C(I,X) \to C(I,X)$ are also continuous functions. We recall also that the derivative D_t^α denotes the Caputo fractional derivative.

The mild solution to problem (4.5) is given by

$$u(t) \;=\; S_{\alpha,1}(t)(u_0 - p(u)) + S_{\alpha,2}(t)(u_1 - q(u)) + \int_0^t S_{\alpha,\alpha}(t-s)f(s,u(s))ds, \quad t\in[0,T].$$

By the uniqueness of the Laplace transform, it is easy to see that the mild solution to the fractional non-local Problem (4.5) can be written as

$$u(t) \;=\; S_{\alpha,1}(t)(u_0-p(u)) + (g_1 * S_{\alpha,1})(t)(u_1 - q(u)) + \int_0^t (g_{\alpha-1} * S_{\alpha,1})(t-s)f(s,u(s))ds, \tag{4.6}$$

for all $t\in[0,T]$.

We assume the following

- **H1.** The function f satisfies the Carathéodory condition, that is $f(\cdot,u)$ is strongly measurable for each $u\in X$ and $f(t,\cdot)$ is continuous for each $t \in I := [0,T]$.
- **H2.** There exists a continuous function $\mu : I \to \mathbb{R}_+$ such that
$$\|f(t,u)\| \leq \mu(t)\|u\|, \quad \forall\, t\in I,\ u \in C(I,X).$$
- **H3.** The functions $p, q : C(I,X)\to C(I,X)$ are continuous and there exist $L_p, L_q > 0$ such that
$$\|p(u)-p(v)\| < L_p\|u-v\|, \quad \|q(u)-q(v)\| < L_q\|u-v\|,\ \forall u,v\in C(I,X).$$

We have the following existence results.

Theorem 4.20. *Let $1<\alpha<2$. Let A be the generator of an $(\alpha,1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ of type (M,ω). Suppose that $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega^{1/\alpha}$. If $\frac{Me^{\omega T}}{\omega^{\alpha-1}}\|\mu\|_\infty T < 1$ and $\left(Me^{\omega T}L_p + \frac{M}{\omega}e^{\omega T}L_q\right) < 1$, then, under assumptions H1-H3, the Problem (4.5) has at least one mild solution.*

Proof. Let $B_r := \{u \in C(I,X) : \|u\| \leq r\}$, where

$$r := \frac{Me^{\omega T}(\|u_0\| + \|p(u)\|) + \frac{M}{\omega}e^{\omega T}(\|u_1\| + \|q(u)\|)}{1 - \frac{Me^{\omega T}}{\omega^{\alpha-1}}\|\mu\|_\infty T}.$$

On B_r we define the operators Γ_1, Γ_2 by

$$\begin{aligned}(\Gamma_1 u)(t): &= S_{\alpha,1}(t)[u_0 - p(u)] + (g_1 * S_{\alpha,1})(t)(u_1 - q(u)) \quad t \in [0,T] \\ (\Gamma_2 u)(t): &= \int_0^t (g_{\alpha-1} * S_{\alpha,1})(t-s) f(s,u(s))ds, \quad t \in [0,T],\end{aligned}$$

and $u \in B_r$. We shall prove that $\Gamma := \Gamma_1 + \Gamma_2$ has at least one fixed point by the Krasnoselskii fixed point theorem. We will consider several steps in the proof.

Step 1. We will see that if $u, v \in B_r$, then $\Gamma_1 u + \Gamma_2 v \in B_r$. In fact, by Lemma 3.12 we have

$$\begin{aligned}&\|(\Gamma_1 u)(t) + (\Gamma_2 v)(t)\| \leq \\ &\leq \|S_{\alpha,1}(t)\| \, \|u_0 - p(u)\| + \|(g_1 * S_{\alpha,1})(t)\| \, \|u_1 - q(u)\| + \\ &\quad \int_0^t \|(g_{\alpha-1} * S_{\alpha,1})(t-s)\| \, \|f(s,v(s))\|ds \\ &\leq Me^{\omega t}(\|u_0\| + \|p(u)\|) + \frac{M}{\omega} e^{\omega t}(\|u_1\| + \|q(u)\|) + \\ &\quad \int_0^t \|(g_{\alpha-1} * S_{\alpha,1})(t-s)\| \|f(s,v(s))\|ds \\ &\leq Me^{\omega T}(\|u_0\| + \|p(u)\|) + \frac{M}{\omega} e^{\omega T}(\|u_1\| + \|q(u)\|) + \frac{M}{\omega^{\alpha-1}} \int_0^t e^{\omega(t-s)} \mu(s) \|v(s)\|ds \\ &\leq Me^{\omega T}(\|u_0\| + \|p(u)\|) + \frac{M}{\omega} e^{\omega T}(\|u_1\| + \|q(u)\|) + \frac{Mre^{\omega t}}{\omega^{\alpha-1}} \int_0^t e^{-\omega s} \mu(s)ds \\ &\leq Me^{\omega T}(\|u_0\| + \|p(u)\|) + \frac{M}{\omega} e^{\omega T}(\|u_1\| + \|q(u)\|) + \frac{Mre^{\omega T}}{\omega^{\alpha-1}} \|\mu\|_\infty T = r.\end{aligned}$$

Hence $\Gamma_1 u + \Gamma_2 v \in B_r$ for all $u, v \in B_r$.

Step 2. Γ_1 is a contraction on B_r. In fact, if $u, v \in B_r$, then

$$\begin{aligned}\|\Gamma_1 u(t) - \Gamma_1 v(t)\| &\leq \|S_{\alpha,1}(t)\| \, \|p(u) - p(v)\| + \|(g_1 * S_{\alpha,1})(t)\| \, \|q(u) - q(v)\| \\ &\leq Me^{\omega t} L_p \|u - v\| + \frac{M}{\omega} e^{\omega t} L_q \|u - v\| \\ &\leq \left(Me^{\omega T} L_p + \frac{M}{\omega} e^{\omega T} L_q \right) \|u - v\|.\end{aligned}$$

Since $\left(Me^{\omega T} L_p + \frac{M}{\omega} e^{\omega T} L_q\right) < 1$, we conclude that Γ_1 is a contraction.

Step 3. Γ_2 is completely continuous.

Firstly, we prove that Γ_2 is a continuous operator on B_r. Let $u_n, u \in B_r$ such that $u_n \to u$ in B_r. By Lemma 3.12 we get

$$\begin{aligned}\|\Gamma_2 u_n(t) - \Gamma_2 u(t)\| &\leq \int_0^t \|(g_{\alpha-1} * S_{\alpha,1})(t-s)\| \|f(s,u_n(s)) - f(s,u(s))\|ds \\ &\leq \frac{Me^{\omega t}}{\omega^{\alpha-1}} \int_0^t e^{-\omega s} \|f(s,u_n(s)) - f(s,u(s))\|ds \\ &\leq \frac{Me^{\omega T}}{\omega^{\alpha-1}} \int_0^t \mu(s)(\|u_n(s)\| + \|u(s)\|)ds \\ &\leq \frac{2rMe^{\omega T}}{\omega^{\alpha-1}} \int_0^t \mu(s)ds.\end{aligned}$$

We notice that the function $s \mapsto \mu(s)$ is integrable on I. By the Lebesgue's Dominated Convergence Theorem, $\int_0^t \|f(s,u_n(s)) - f(s,u(s))\|ds \to 0$ as $n \to \infty$. Since $u_n \to u$ we obtain that Γ_2 is continuous in B_r.

Now, we will prove that $\{\Gamma_2 u : u \in B_r\}$ is relatively compact. By the Ascoli-Arzela theorem we need to show that the family $\{\Gamma_2 u : u \in B_r\}$ is uniformly bounded and equicontinuous, and the set $\{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for each $t \in [0,T]$. In fact, for each $u \in B_r$ we have (as in Step 3) that $\|\Gamma_2 u\| \le \frac{rMe^{\omega T}}{\omega^{\alpha-1}}\|\mu\|_\infty$ and therefore $\{\Gamma_2 u : u \in B_r\}$ is uniformly bounded.

In order to prove the equicontinuity, let $u \in B_r$, and take $0 \le t_2 < t_1 \le T$. Observe that

$$\begin{aligned} \|\Gamma_2 u(t_1) - \Gamma_2 u(t_2)\| &\le \int_{t_2}^{t_1} \|(g_{\alpha-1} * S_{\alpha,1})(t_1 - s) f(s,u(s))\|ds \\ &+ \int_0^{t_2} \|\left((g_{\alpha-1} * S_{\alpha,1})(t_1 - s) - (g_1 * S_{\alpha,1})(t_2 - s)\right) f(s,u(s))\|ds \\ &:= I_1 + I_2. \end{aligned}$$

Observe that for I_1, by Lemma 3.12 we have

$$I_1 \le \frac{Me^{\omega T}}{\omega^{\alpha-1}} \int_{t_2}^{t_1} e^{-\omega s}\mu(s)\|u(s)\|ds \le \frac{Mre^{\omega T}}{\omega^{\alpha-1}}\|\mu\|_\infty (t_1 - t_2),$$

and therefore $\lim_{t_1 \to t_2} I_1 = 0$. For I_2 we have

$$\begin{aligned} I_2 &\le \int_0^{t_2} \|(g_{\alpha-1} * S_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S_{\alpha,1})(t_2 - s)\|\|f(s,u(s))\|ds \\ &\le \int_0^{t_2} \mu(s)\|(g_{\alpha-1} * S_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S_{\alpha,1})(t_2 - s)\|\,\|u(s)\|ds \\ &\le r\int_0^{t_2} \mu(s)\|(g_{\alpha-1} * S_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S_{\alpha,1})(t_2 - s)\|ds. \end{aligned}$$

Observe that

$$\mu(\cdot)\|(g_{\alpha-1} * S_{\alpha,1})(t_1 - \cdot) - (g_{\alpha-1} * S_{\alpha,1})(t_2 - \cdot)\| \le 2\frac{Me^{\omega T}}{\omega^{\alpha-1}}\mu(\cdot) \in L^1(I,\mathbb{R}),$$

and by Lemma 3.12, $(g_{\alpha-1} * S_{\alpha,1})(t) = S_{\alpha,\alpha}(t)$ for all $t \ge 0$. Moreover, by Proposition 3.11 we have that $S_{\alpha,\alpha}(t)$ is norm continuous and therefore if $t_1 \to t_2$, then $(g_{\alpha-1} * S_{\alpha,1})(t_1 - s) - (g_{\alpha-1} * S_{\alpha,1})(t_2 - s) \to 0$ in $\mathcal{B}(X)$. We obtain by the Lebesgue's dominated convergence theorem that $\lim_{t_1 \to t_2} I_2 = 0$. Therefore, $\{\Gamma_2 u : u \in B_r\}$ is an equicontinuous family.

Finally, we prove that $H(t) := \{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for each $t \in [0,T]$. Obviously, $H(0)$ is relatively compact in X. Now, we take $t > 0$. For $0 < \varepsilon < t$ we define on B_r the operator

$$(\Gamma_2^\varepsilon u)(t) : \;=\; \int_0^{t-\varepsilon} (g_{\alpha-1} * S_{\alpha,1})(t - s) f(s,u(s))ds.$$

The hypotheses implies the compactness of $(g_{\alpha-1} * S_{\alpha,1})(t) = S_{\alpha,\alpha}(t)$ for all $t > 0$ (by Lemma 3.12 and by Theorem 3.14) and therefore the set $\mathcal{K}_\varepsilon := \{(g_{\alpha-1} * S_{\alpha,1})(t - s) f(s,u(s)) : u \in B_r, 0 \le s \le t - \varepsilon\}$ is compact for all $\varepsilon > 0$. Then $\overline{\mathrm{conv}(\mathcal{K}_\varepsilon)}$ is also a compact set by Theorem 2.7. By using the Mean-Value Theorem for the Bochner integrals (see [8, Corollary 8, p. 48]), we obtain that

$$(\Gamma_2^\varepsilon u)(t) \in t\overline{\mathrm{conv}(\mathcal{K}_\varepsilon)}, \text{ for all } t \in [0,T].$$

Therefore, the set $H_\varepsilon(t) := \{(\Gamma_2^\varepsilon u)(t) : u \in B_r\}$ is relatively compact in X for all $\varepsilon > 0$. Now, observe that

$$\begin{aligned} \|(\Gamma_2 u)(t) - (\Gamma_2^\varepsilon u)(t)\| &\leq \int_{t-\varepsilon}^t \|(g_{\alpha-1} * S_{\alpha,1})(t-s) f(s,u(s))\| ds \\ &\leq \frac{Mre^{\omega T}}{\omega^{\alpha-1}} \int_{t-\varepsilon}^t e^{-\omega s}\mu(s) ds \end{aligned}$$

Since the function $s \mapsto e^{-\omega s}\mu(s)$ belongs to $L^1([t-\varepsilon, t], \mathbb{R}_+)$ we conclude by the Lebesgue dominated convergence Theorem that

$$\lim_{\varepsilon \to 0} \|(\Gamma_2 u)(t) - (\Gamma_2^\varepsilon u)(t)\| = 0.$$

Therefore the set $\{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for each $t \in (0, T]$. By the Ascoli-Arzela theorem, the set $\{\Gamma_2 u : u \in B_r\}$ is relatively compact. We conclude that Γ_2 is a completely continuous operator. Hence, by the Krasnoselskii Theorem 2.8 we have that $\Gamma = \Gamma_1 + \Gamma_2$ has a fixed point on B_r, which means that the nonlocal problem (4.5) has a mild solution and the proof of the Theorem is finished. □

The proof of the following result uses the Schauder fixed point theorem. We notice that here we will assume that $S_{\alpha,1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Moreover, we have a weaker condition on the parameters M, ω and T.

Theorem 4.21. *Let $1 < \alpha < 2$. Let A be the generator of an $(\alpha, 1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t \geq 0}$ of type (M, ω). Suppose that $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega^{1/\alpha}$, $S_{\alpha,1}(t)$ is continuous in the uniform operator topology for all $t > 0$, and $Me^{\omega T}\|\mu\|_\infty T < 1$. Then, under assumptions H1-H3, the Problem (4.5) has at least one mild solution.*

Proof. We define the operator $\Gamma : C(I, X) \to C(I, X)$ by

$$\begin{aligned} (\Gamma u)(t) : &= S_{\alpha,1}(t)[u_0 - p(u)] + (g_1 * S_{\alpha,1})(t)(u_1 - q(u)) \\ &+ \int_0^t (g_{\alpha-1} * S_{\alpha,1})(t-s) f(s, u(s)) ds, \quad t \in I = [0, T]. \end{aligned}$$

Choose

$$r = \frac{Me^{\omega T}(\|u_0\| + \|p(u)\|) + \frac{M}{\omega} e^{\omega T}(\|u_1\| + \|q(u)\|)}{1 - \frac{Me^{\omega T}}{\omega^{\alpha-1}}\|\mu\|_\infty T}.$$

Let $B_r := \{u \in C(I, X) : \|u\| \leq r\}$. We shall prove that $\Gamma : B_r \to B_r$ has at least one fixed point by the Schauder fixed point theorem. As in the proof of Theorem 4.20 it is easy to see that Γ sends B_r into B_r, and $\Gamma : B_r \to B_r$ is a continuous operator.

We claim that $\{\Gamma u : u \in B_r\}$ is relatively compact.

Indeed, as in the proof of Theorem 4.20 it is easy to see that $\{\Gamma u : u \in B_r\}$ is uniformly bounded. On the other hand, to see the equicontinuity, let $u \in B_r$, and take $t_1, t_2 \in I$ with $0 \leq t_2 < t_1 \leq T$. We have

$$\begin{aligned} &\|\Gamma u(t_1) - \Gamma u(t_2)\| \leq \\ &\leq \|\left(S_{\alpha,1}(t_1) - S_{\alpha,1}(t_2)\right)(u_0 - p(u))\| + \|\left((g_1 * S_{\alpha,1})(t_1) - (g_1 * S_{\alpha,1})(t_2)\right)(u_1 - q(u))\| \\ &+ \int_{t_2}^{t_1} \|(g_{\alpha-1} * S_{\alpha,1})(t_1 - s) f(s, u(s))\| ds \\ &+ \int_0^{t_2} \|\left((g_{\alpha-1} * S_{\alpha,1})(t_1 - s) - (g_1 * S_{\alpha,1})(t_2 - s)\right) f(s, u(s))\| ds \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Observe that for I_1 we have

$$I_1 \leq \|S_{\alpha,1}(t_1) - S_{\alpha,1}(t_2)\| \|(u_0 - g(u))\|.$$

By hypothesis, using the norm continuity of $S_{\alpha,1}(t)$, we obtain that $\lim_{t_1 \to t_2} I_1 = 0$.

The Lemma 3.12 implies $(g_1 * S_{\alpha,1})(t) = S_{\alpha,2}(t)$ for all $t \geq 0$ and by Proposition 3.11 we have that $(g_1 * S_{\alpha,1})(t)$ is continuous in $\mathcal{B}(X)$, and hence

$$I_2 \leq \|(g_1 * S_{\alpha,1})(t_1) - (g_1 * S_{\alpha,1})(t_2)\| \|(u_0 - g(u))\| \to 0,$$

as $t_1 \to t_2$. On the other hand, $I_3, I_4 \to 0$ as $t_1 \to t_2$ as in the proof of Step 3 in Theorem 4.20. Therefore, the set $\{\Gamma u : u \in B_r\}$ is equicontinuous.

Finally, we will prove that $\{\Gamma u(t) : u \in B_r\}$ is relatively compact for all $t \in [0, T]$. Clearly, $\{\Gamma u(0) : u \in B_r\}$ is relatively compact. Now, we take $t > 0$. For each $0 < \varepsilon < t$ we define the operator

$$(\Gamma^{\varepsilon} u)(t) : \quad = \quad S_{\alpha,1}(\varepsilon) \int_0^{t-\varepsilon} (g_{\alpha-1} * S_{\alpha,1})(t - s - \varepsilon) f(s, u(s)) ds.$$

The hypothesis and Proposition 3.16 show that $S_{\alpha,1}(t)$ is compact for all $t > 0$ and therefore the set $H_{\varepsilon}(t) := \{(\Gamma_2^{\varepsilon} u)(t) : u \in B_r\}$ is relatively compact in X for all $\varepsilon > 0$. Now, observe that

$$\begin{aligned} & \left\| S_{\alpha,1}(\varepsilon) \int_0^{t-\varepsilon} (g_{\alpha-1} * S_{\alpha,1})(t - s - \varepsilon) f(s, u(s)) ds - \int_0^{t-\varepsilon} (g_{\alpha-1} * S_{\alpha,1})(t - s) f(s, u(s)) ds \right\| \\ \leq \quad & r \int_0^{t-\varepsilon} \|S_{\alpha,1}(\varepsilon)(g_{\alpha-1} * S_{\alpha,1})(t - s - \varepsilon) - (g_{\alpha-1} * S_{\alpha,1})(t - s)\| \mu(s) ds. \end{aligned}$$

By Proposition 3.11, $(g_{\alpha-1} * S_{\alpha,1})(t)$ is norm continuous for all $t > 0$ and therefore

$$\|S_{\alpha,1}(\varepsilon)(g_{\alpha-1} * S_{\alpha,1})(t - s - \varepsilon) - (g_{\alpha-1} * S_{\alpha,1})(t - s)\| \to 0, \text{ as } \varepsilon \to 0.$$

On the other hand, since

$$\|S_{\alpha,1}(\varepsilon)(g_{\alpha-1} * S_{\alpha,1})(t - \cdot - \varepsilon) - (g_{\alpha-1} * S_{\alpha,1})(t - \cdot)\| \quad \leq \quad \frac{M^2 e^{2\omega T}}{\omega^{\alpha-1}} e^{-\omega(\cdot + \varepsilon)} + \frac{M e^{\omega T}}{\omega^{\alpha-1}} e^{-\omega \cdot},$$

and the function $s \mapsto \frac{M^2 e^{2\omega T}}{\omega^{\alpha-1}} e^{-\omega(s+\varepsilon)} + \frac{M e^{\omega T}}{\omega^{\alpha-1}} e^{-\omega s}$ belongs to $L^1(I, \mathbb{R}_+)$ we conclude by the Lebesgue dominated convergence Theorem that

$$\lim_{\varepsilon \to 0} \left\| S_{\alpha,1}(\varepsilon) \int_0^{t-\varepsilon} (g_{\alpha-1} * S_{\alpha,1})(t - s - \varepsilon) f(s, u(s)) ds - \int_0^{t-\varepsilon} (g_{\alpha-1} * S_{\alpha,1})(t - s) f(s, u(s)) ds \right\| = 0.$$

As in the proof of [9, Theorem 4.1] we get

$$\lim_{\varepsilon \to 0} \left\| S_{\alpha,1}(\varepsilon) \int_0^{t-\varepsilon} (g_{\alpha-1} * S_{\alpha,1})(t - s - \varepsilon) f(s, u(s)) ds - \int_0^{t} (g_{\alpha-1} * S_{\alpha,1})(t - s) f(s, u(s)) ds \right\| = 0,$$

and therefore the set $\{\int_0^t (g_{\alpha-1} * S_{\alpha,1})(t - s) f(s, u(s)) ds : u \in B_r\}$ is relatively compact for all $t \in (0, T]$. The compactness of $S_{\alpha,1}(t)$ and $(g_1 * S_{\alpha,1})(t) = S_{\alpha,2}(t)$ (by Lemma 3.12 and Theorem 3.14) imply that $\{\Gamma u(t) : u \in B_r\}$ is relatively compact in X for each $t \in (0, T]$. By the Ascoli-Arzela theorem, the set $\{\Gamma u : u \in B_r\}$ is relatively compact. We conclude that Γ is a compact operator on B_r. Hence, by the Schauder (Theorem 2.9) we have that Γ has a fixed point on B_r and therefore the nonlocal problem (4.5) has a mild solution. □

Remark 4.22. *We notice that the norm continuity of $S_{\alpha,1}(t)$ for $0 < \alpha < 1$ and $t > 0$ it follows, for example, if $\{S_{\alpha,1}(t)\}_{t \geq 0}$ is analytic (see* [9, Lemma 3.8]*), or if A is an almost sectorial operator (see* [26, Theorem 3.2]*).*

Now, we consider the non-local problem for the Caputo fractional derivative

$$\left\{\begin{array}{lcl} D_t^\alpha u(t) & = & Au(t)+f(t,u(t)), \quad t\in I:=[0,T] \\ u(0)+p(u) & = & u_0 \end{array}\right. \tag{4.7}$$

$u_0 \in X$, $1/2<\alpha<1$, $T>0$ and A is a closed linear operator defined on X which generates the (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$.

The mild solution to problem (4.7) is given by

$$u(t) \;=\; S_{\alpha,1}(t)(u_0-p(u))+\int_0^t S_{\alpha,\alpha}(t-s)f(s,u(s))ds, \quad t\in[0,T].$$

It is easy to see (by using the uniqueness of the Laplace transform) that the mild solution to the problem (4.7) can be also written as

$$u(t) \;=\; (g_{1-\alpha}*S_{\alpha,\alpha})(t)(u_0-p(u))+\int_0^t S_{\alpha,\alpha}(t-s)f(s,u(s))ds, \quad t\in[0,T]. \tag{4.8}$$

The proof of the following result follows similarly to Theorem 4.20 and therefore, we omit it.

Theorem 4.23. *Let $1/2<\alpha<1$. Let A be the generator of an (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ of type (M,ω). Suppose that $(\lambda^\alpha-A)^{-1}$ is compact for all $\lambda>\omega^{1/\alpha}$, and $S_{\alpha,\alpha}(t)$ is continuous in the uniform operator topology for all $t>0$. If $Me^{\omega T}\|\mu\|_\infty T<1$, and $\frac{M}{\omega^{1-\alpha}}e^{\omega T}L_p<1$, then, under assumptions H1-H3, the Problem (4.7) has at least one mild solution.*

5. Non-local fractional Cauchy problems. The Riemann-Liouville case.

In this section we consider the non-local problem for the Riemann-Liouville fractional derivative

$$\left\{\begin{array}{lcl} D^\alpha u(t) & = & Au(t)+f(t,u(t)), \quad t\in[0,T] \\ (g_{2-\alpha}*u)(0)+p(u) & = & u_0 \\ (g_{2-\alpha}*u)'(0)+q(u) & = & u_1, \end{array}\right. \tag{5.9}$$

where $u_0,u_1\in X$, $1<\alpha<2$ and A is a closed linear operator defined on X. Assume that A generates an $(\alpha,\alpha-1)$-resolvent family given by $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$. Taking Laplace transform in (5.9) we obtain by (2.1) that

$$u(t)=S_{\alpha,\alpha-1}(t)(u_0-p(u))+S_{\alpha,\alpha}(t)(u_1-q(u))+\int_0^t S_{\alpha,\alpha}(t-s)f(s,u(s))ds, t\in[0,T].$$

The uniqueness of the Laplace transform implies that the mild solution u to problem (5.9) is also given by

$$u(t)=S_{\alpha,\alpha-1}(t)(u_0-p(u))+(g_1*S_{\alpha,\alpha-1})(t)(u_1-q(u))+\int_0^t (g_1*S_{\alpha,\alpha-1})(t-s)f(s,u(s))ds,$$

for all $t\in[0,T]$.

Theorem 5.24. *Let $1<\alpha<2$. Let A be the generator of an $(\alpha,\alpha-1)$-resolvent family $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ of type (M,ω). Assume that the resolvent $(\lambda^\alpha-A)^{-1}$ is compact for all $\lambda>\omega^{1/\alpha}$. If $\left(Me^{\omega T}L_p+\frac{M}{\omega}e^{\omega T}L_q\right)<1$ and $\frac{Me^{\omega T}}{\omega}\|\mu\|_\infty T<1$, then, under assumptions H1-H3, the Problem (5.9) has at least one mild solution.*

Proof. Let $B_r:=\{u\in C(I,X):\|u\|\leq r\}$, where

$$r:=\frac{Me^{\omega T}(\|u_0\|+\|p(u)\|)+\frac{M}{\omega}e^{\omega T}(\|u_1\|+\|q(u)\|)}{1-\frac{Me^{\omega T}}{\omega}\|\mu\|_\infty T}.$$

On B_r we define the operators Γ_1, Γ_2 by

$$\begin{aligned}(\Gamma_1 u)(t): &= S_{\alpha,\alpha-1}(t)[u_0 - p(u)] + (g_1 * S_{\alpha,\alpha-1})(t)(u_1 - q(u)) \quad t \in [0,T]\\ (\Gamma_2 u)(t): &= \int_0^t (g_1 * S_{\alpha,\alpha-1})(t-s) f(s,u(s))ds, \quad t \in [0,T],\end{aligned}$$

and $u \in B_r$. We shall prove that $\Gamma := \Gamma_1 + \Gamma_2$ has at least one fixed point by the Krasnoselskii fixed point theorem. We will consider several steps in the proof.

Step 1. We will see that if $u, v \in B_r$, then $\Gamma_1 u + \Gamma_2 v \in B_r$. In fact, by Lemma 3.12 we have

$$\begin{aligned}&\|(\Gamma_1 u)(t) + (\Gamma_2 v)(t)\| \leq\\ &\leq \|S_{\alpha,\alpha-1}(t)\| \, \|u_0 - p(u)\| + \|(g_1 * S_{\alpha,\alpha-1})(t)\| \, \|u_1 - q(u)\| + \\ &\quad \int_0^t \|(g_1 * S_{\alpha,\alpha-1})(t-s)\| \, \|f(s,v(s))\| ds\\ &\leq Me^{\omega t}(\|u_0\| + \|p(u)\|) + \frac{M}{\omega} e^{\omega t}(\|u_1\| + \|q(u)\|) + \frac{Mre^{\omega t}}{\omega} \int_0^t e^{-\omega s}\mu(s) ds\\ &\leq Me^{\omega T}(\|u_0\| + \|p(u)\|) + \frac{M}{\omega} e^{\omega T}(\|u_1\| + \|q(u)\|) + \frac{Mre^{\omega T}}{\omega} \|\mu\|_\infty T = r.\end{aligned}$$

Hence $\Gamma_1 u + \Gamma_2 v \in B_r$ for all $u, v \in B_r$.

Step 2. Γ_1 is a contraction on B_r. In fact, if $u, v \in B_r$, then

$$\begin{aligned}\|\Gamma_1 u(t) - \Gamma_1 v(t)\| &\leq \|S_{\alpha,\alpha-1}(t)\| \, \|p(u) - p(v)\| + \|(g_1 * S_{\alpha,\alpha-1})(t)\| \, \|q(u) - q(v)\|\\ &\leq Me^{\omega t} L_p \|u - v\| + \frac{M}{\omega} e^{\omega t} L_q \|u - v\|\\ &\leq \left(Me^{\omega T} L_p + \frac{M}{\omega} e^{\omega T} L_q \right) \|u - v\|.\end{aligned}$$

Since $\left(Me^{\omega T} L_p + \frac{M}{\omega} e^{\omega T} L_q\right) < 1$, we conclude that Γ_1 is a contraction.

Step 3. Γ_2 is completely continuous.

As in the proof of Theorem 4.20 it is easy to see that Γ_2 is a continuous operator and the set $\{\Gamma_2 u : u \in B_r\}$ is uniformly bounded.

To prove the equicontinuity, let $u \in B_r$, and take $0 \leq t_2 < t_1 \leq T$. Observe that

$$\begin{aligned}\|\Gamma_2 u(t_1) - \Gamma_2 u(t_2)\| &\leq \int_{t_2}^{t_1} \|(g_1 * S_{\alpha,\alpha-1})(t_1 - s) f(s,u(s))\| ds\\ &+ \int_0^{t_2} \| \left((g_1 * S_{\alpha,\alpha-1})(t_1 - s) - (g_1 * S_{\alpha,\alpha-1})(t_2 - s) \right) f(s,u(s))\| ds\\ &:= I_1 + I_2.\end{aligned}$$

To estimate I_1 we notice that

$$I_1 \leq \frac{Me^{\omega T}}{\omega} \int_{t_2}^{t_1} e^{-\omega s}\mu(s)\|u(s)\| ds \leq \frac{Mre^{\omega T}}{\omega} \|\mu\|_\infty (t_1 - t_2),$$

and therefore $\lim_{t_1 \to t_2} I_1 = 0$. For I_2 we have

$$\begin{aligned}I_2 &\leq \int_0^{t_2} \|(g_1 * S_{\alpha,\alpha-1})(t_1 - s) - (g_1 * S_{\alpha,\alpha-1})(t_2 - s)\| \|f(s,u(s))\| ds\\ &\leq \int_0^{t_2} \mu(s) \|(g_1 * S_{\alpha,\alpha-1})(t_1 - s) - (g_1 * S_{\alpha,\alpha-1})(t_2 - s)\| \, \|u(s)\| ds\\ &\leq r \int_0^{t_2} \mu(s) \|(g_1 * S_{\alpha,\alpha-1})(t_1 - s) - (g_1 * S_{\alpha,\alpha-1})(t_2 - s)\| ds.\end{aligned}$$

Observe that

$$\mu(\cdot)\|(g_1 * S_{\alpha,\alpha-1})(t_1-\cdot)-(g_1 * S_{\alpha,\alpha-1})(t_2-\cdot)\| \le 2\frac{Me^{\omega T}}{\omega}\mu(\cdot) \in L^1(I,\mathbb{R}),$$

and by Lemma 3.12, $(g_1 * S_{\alpha,\alpha-1})(t) = S_{\alpha,\alpha}(t)$ for all $t \ge 0$. Moreover, by Proposition 3.11 we have that $S_{\alpha,\alpha}(t)$ is norm continuous and therefore if $t_1 \to t_2$, then $(g_1 * S_{\alpha,\alpha-1})(t_1 - s) - (g_1 * S_{\alpha,\alpha-1})(t_2 - s) \to 0$ in $\mathcal{B}(X)$. We obtain by the Lebesgue's dominated convergence theorem that $\lim_{t_1\to t_2} I_2 = 0$. Therefore, $\{\Gamma_2 u : u \in B_r\}$ is an equicontinuous family.

Finally, the compactness of $(g_1 * S_{\alpha,\alpha-1})(t) = S_{\alpha,\alpha}(t)$ for all $t > 0$ (by Lemma 3.12 and Theorem 3.14) implies that $\{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for each $t \in [0,T]$ (as in the proof of Theorem 4.20). We conclude that Γ_2 is a completely continuous operator and by the Krasnoselskii Theorem, the operator $\Gamma = \Gamma_1 + \Gamma_2$ has a fixed point on B_r, which means that the nonlocal problem (5.9) has at least one mild solution. □

In the next result, we consider a weaker condition on the parameters M, ω and T. However, we need to assume here the norm continuity of $S_{\alpha,\alpha-1}(t)$ for $3/2 < \alpha < 2$.

Theorem 5.25. *Let $3/2 < \alpha < 2$. Let A be the generator of an $(\alpha,\alpha-1)$-resolvent family $\{S_{\alpha,\alpha-1}(t)\}_{t\ge0}$ of type (M,ω). Assume that $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega^{1/\alpha}$ and $S_{\alpha,\alpha-1}(t)$ is continuous in the uniform operator topology for all $t > 0$. If $\frac{Me^{\omega T}}{\omega}\|\mu\|_\infty T < 1$, then, under assumptions H1-H3, the Problem (5.9) has at least one mild solution.*

Proof. On B_r we define the operator

$$\Gamma u(t) := S_{\alpha,\alpha-1}(t)(u_0 - p(u)) + (g_1 * S_{\alpha,\alpha-1})(t)(u_1 - q(u)) + \int_0^t (g_1 * S_{\alpha,\alpha-1})(t-s)f(s,u(s))ds,$$

where $t \in [0,T]$ and

$$r := \frac{Me^{\omega T}(\|u_0\| + \|p(u)\|) + \frac{M}{\omega}e^{\omega T}(\|u_1\| + \|q(u)\|)}{1 - \frac{Me^{\omega T}}{\omega}\|\mu\|_\infty T}.$$

The proof follows the same lines of Theorem 4.21. We give here only the details on the relatively compactness of $\{\Gamma_2 u(t) : u \in B_r\}$ in X for each $t \in [0,T]$. The Theorem 3.14 implies that $(g_1 * S_{\alpha,\alpha-1})(t) = S_{\alpha,\alpha}(t)$ is compact for all $t > 0$ and therefore the set $\{\int_0^t (g_1 * S_{\alpha,\alpha-1})(t-s)f(s,u(s))ds : u \in B_r\}$ is relatively compact for all $t \in [0,T]$ (as in the proof of Theorem 4.20). On the other hand, the hypothesis and Proposition 3.17 imply that $S_{\alpha,\alpha-1}(t)$ is compact for all $t > 0$ and thus the set $\{\Gamma u(t) : u \in B_r\}$ is relatively compact for all $t \in [0,T]$. The existence of a fixed point to Γ, and therefore of a mild solution to problem (5.9), follows from the Schauder Theorem. □

Now we discuss the existence of mild solutions to the nonlocal fractional Cauchy problem for the Riemann-Liouville fractional derivative in case $0 < \alpha < 1$

$$\left\{\begin{array}{rcll} D^\alpha u(t) &=& Au(t) + f(t,u(t)), & t \in [0,T] \\ (g_{1-\alpha} * u)(0) + p(u) &=& u_0 & \end{array}\right. \tag{5.10}$$

where $u_0 \in X$, and A is a closed linear operator defined on X. We assume that A generates an (α,α)-resolvent family given by $\{S_{\alpha,\alpha}(t)\}_{t\ge0}$. By using the Laplace transform in (5.10) it is easy to see that

$$u(t) = S_{\alpha,\alpha}(t)(u_0 - p(u)) + \int_0^t S_{\alpha,\alpha}(t-s)f(s,u(s))ds, \quad t \in [0,T].$$

Theorem 5.26. *Let $1/2 < \alpha < 1$. Let A be the generator of an (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\ge0}$ of type (M,ω). Assume that $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega^{1/\alpha}$, and $S_{\alpha,\alpha}(t)$ is continuous in the uniform operator topology for all $t > 0$. If $Me^{\omega T}\|\mu\|_\infty T < 1$ and $Me^{\omega T}L_p < 1$, then, under assumptions H1-H3, the Problem (5.10) has at least one mild solution.*

Proof. Let
$$r := \frac{Me^{\omega T}(\|u_0\| + \|p(u)\|)}{1 - Me^{\omega T}\|\mu\|_\infty T}.$$
If we define on B_r the operators Γ_1, Γ_2 by
$$\begin{aligned}(\Gamma_1 u)(t) : &= S_{\alpha,\alpha}(t)[u_0 - p(u)] \quad t \in [0,T]\\ (\Gamma_2 u)(t) : &= \int_0^t S_{\alpha,\alpha}(t-s)f(s,u(s))ds, \quad t \in [0,T],\end{aligned}$$
for $u \in B_r$, then as in the proof of the previous theorems, it is easy to see that if $u, v \in B_r$, then $\Gamma_1 u + \Gamma_2 v \in B_r$, and Γ_1 is a contraction on B_r. Moreover, Γ_2 is continuous on B_r, $\{\Gamma_2 u : u \in B_r\}$ is uniformly bounded and $\{\Gamma_2 u : u \in B_r\}$ is an equicontinuous family. Finally, the compactness of $S_{\alpha,\alpha}(t)$ (see Proposition 3.18) and by using a similar method as we did in the in the proof Theorem 4.20 (Step 3) we prove that $H(t) := \{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for each $t \in [0,T]$. Thus, by the Ascoli-Arzela theorem, the set $\{\Gamma_2 u : u \in B_r\}$ is relatively compact and hence Γ_2 is a completely continuous operator. By the Krasnoselskii Theorem we conclude that $\Gamma = \Gamma_1 + \Gamma_2$ has a fixed point on B_r, and therefore the nonlocal problem (5.10) has at least one mild solution. □

6. Applications

In this section, we give some applications. As consequence of the previous results we have the following results.

Consider the semilinear problem

$$(6.11) \qquad \begin{cases} D_t^\alpha u(t) &= Au(t) + J^{2-\alpha}f(t,u(t)), \quad t \in I := [0,T]\\ u(0) + p(u) &= u_0\\ u'(0) + q(u) &= u_1,\end{cases}$$

where $u_0, u_1 \in X$, $J^{2-\alpha}$ denotes the Riemann-Liouville fractional integral operator, $f : [0,T] \times X \to X$ and $p, q : C(I,X) \to C(I,X)$ are continuous.

Let A be the generator of an $(\alpha,1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$. Then it is well known that the mild solution of (6.11) is defined by means of the variation-of-constant formula

$$u(t) = S_{\alpha,1}(t)[u_0 - p(u)] + (g_1 * S_{\alpha,1})(t)[u_1 - q(u)] + \int_0^t (g_1 * S_{\alpha,1})(t-s)f(s,u(s))ds, \quad t \in I.$$

We remark that the case $0 < \alpha < 1$ was recently studied in [18, Section 4]. On the other hand, we notice that the case $u'(0) = 0$ and $q \equiv 0$ has been recently studied in [17, Section 4] by assuming the relatively compactness of the set $\mathcal{K} := \{S_{\alpha,1}(t-s)f(s,u(s)) : u \in C(I,X), 0 \leq s \leq t\}$. The Proposition 3.16 shows that $S_{\alpha,1}(t)$ is compact for all $t > 0$ and by using the Leray-Schauder Alternative Theorem (see Theorem 2.10) it is easy to prove (as in Theorem 4.21 and [17, Theorem 4.4]) the following result. We omit the details.

Theorem 6.27. *Let $1 < \alpha < 2$. Let A be the generator of an $(\alpha,1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ of type (M,ω). Suppose that $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega^{1/\alpha}$, $S_{\alpha,1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Then, under assumptions H1-H3, the Problem (6.11) has at least one mild solution.*

Now, we consider the Riemann-Liouville fractional Cauchy problem

$$(6.12) \qquad \begin{cases} D^\alpha u(t) &= Au(t) + J^{2-\alpha}f(t,u(t)), \quad t \in [0,T]\\ (g_{2-\alpha} * u)(0) + p(u) &= u_0\\ (g_{2-\alpha} * u)'(0) + q(u) &= u_1,\end{cases}$$

where $u_0, u_1 \in X$, $1 < \alpha < 2$ and A is a closed linear operator defined on X. Assume that A generates an $(\alpha, \alpha-1)$-resolvent family given by $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$. The mild solution to problem (6.12) is given by

$$u(t) = S_{\alpha,\alpha-1}(t)(u_0 - p(u)) + S_{\alpha,\alpha}(t)(u_1 - q(u)) + \int_0^t S_{\alpha,2}(t-s)f(s,u(s))ds, t \in [0,T],$$

which is equivalent (by the uniqueness of the Laplace transform) to

$$u(t) = S_{\alpha,\alpha-1}(t)(u_0 - p(u)) + (g_1 * S_{\alpha,\alpha-1})(t)(u_1 - q(u)) + \int_0^t (g_{3-\alpha} * S_{\alpha,\alpha-1})(t-s)f(s,u(s))ds,$$

for all $t \in [0,T]$.

The Proposition 3.17 shows that $S_{\alpha,\alpha-1}(t)$ is compact for all $t > 0$ (and $3/2 < \alpha < 2$) and by using the Leray-Schauder Alternative Theorem it is easy to prove (as in Theorem 5.25 and [17, Theorem 4.4],[18, Theorem 4.1]) the following existence result. We omit the proof.

Theorem 6.28. *Let $3/2 < \alpha < 2$. Let A be the generator of an $(\alpha, \alpha-1)$-resolvent family $\{S_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ of type (M,ω). Assume that the resolvent $(\lambda^\alpha - A)^{-1}$ is compact for all $\lambda > \omega^{1/\alpha}$ and $S_{\alpha,\alpha-1}(t)$ is continuous in the uniform operator topology for all $t > 0$. Then, under assumptions H1-H3, the Problem (6.12) has at least one mild solution.*

We end this section with an example.

Example 6.29.

Consider the following problem

$$(6.13)\qquad \begin{cases} D_t^\alpha u(t,x) = \dfrac{\partial^2}{\partial x^2} u(t,x) + f(t,u(t,x)), & (t,x) \in [0,1] \times [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \in [0,1], \\ u(0,x) + \sum_{k=1}^n a_k u(t,x) = u_0(x), & x \in [0,\pi], \end{cases}$$

where $1/2 < \alpha < 1$, $a_k \in \mathbb{R}$, $n \in \mathbb{N}$. Let $X = L^2([0,\pi])$ and consider the operator $A : D(A) \subset X \to X$ defined by $D(A) := \{v \in X : v \in H^2([0,\pi]), v(0) = v(\pi)\}$ and for $u \in D(A)$, $Au := \frac{\partial^2 u}{\partial x^2}$.

It is well known that A generates a compact and analytic (and hence norm continuous for all $t > 0$) C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X such that $\|T(t)\| \leq 1$ for all $t \geq 0$. Since A generates a C_0-semigroup, that is, an $(1,1)$-resolvent family, we obtain by [1, Corollary 14 and Theorem 3] that A generates the (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ defined by

$$S_{\alpha,\alpha}(t)x := \int_0^\infty \varphi_{\alpha,0}(t,s)T(s)xds, t > 0, x \in X,$$

where $\varphi_{\alpha,0}$ is the stable Lévy process of order α defined by (2.3). Since $T(t)$ is norm continuous, it is easy to see that $S_{\alpha,\alpha}(t)$ is norm continuous for all $t > 0$ and the positivity of $\varphi_{\alpha,0}$ (see [1, Theorem 3]) implies that $S_{\alpha,\alpha}(t)$ is of type $(1,1)$. On the other hand, the compactness of $T(t)$ implies that $(\lambda^\alpha - A)$ is compact.

We notice that the problem (6.13) can be written in the abstract form (4.7). Define the functions $f : [0,1] \times D(A) \to X$ and $p : D(A) \to X$ by

$$f(t,u(t,x)) := \frac{e^{-t}u(t,x)}{(4+t)(1+u(t,x))}, \quad p(u)(x) := \sum_{k=1}^n a_k u(t,x).$$

Assume that $\sum_{k=1}^n |a_k| < \frac{1}{4}$. We observe also that in this case we have $\mu(t) = \frac{e^{-t}}{4+t}$, $T = M = \omega = 1$, and $L_p = \|\mu\|_\infty = 1/4$ (see the notation in Theorem 4.23).

It is easy to check the assumptions *H1-H3* and the hypotheses in Theorem 4.23 and therefore, the problem (6.13) has a mild solution.

Analogously, we can consider the Riemann-Liouville case

$$(6.14)\qquad \begin{cases} D^\alpha u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + f(t,u(t,x)), \quad (t,x)\in[0,1]\times[0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t\in[0,1], \\ (g_{1-\alpha}*u)(0,x) + \sum_{k=1}^n a_k u(t,x) = u_0(x), \quad x\in[0,\pi]. \end{cases}$$

Under the same assumptions, we have by Theorem 5.26 that the problem (6.14) has a mild solution.

6.1. **Conclusions.** In this paper, we obtain conditions implying the compactness of the family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$. As consequence, we obtain several results on the existence of mild solutions to nonlocal fractional Cauchy problems to the Caputo and Riemann-Liouville fractional derivatives.

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