TIME DISCRETIZATION OF FRACTIONAL SUBDIFFUSION EQUATIONS VIA FRACTIONAL RESOLVENT OPERATORS.

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Abstract. In this work, we study time discretization of subdiffusion equations, that is, fractional differential equations of order $\alpha \in (0,1)$. Assuming that A is the generator of a fractional resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$, which allows to write the solution to the subdiffusion equation $\partial_t^\alpha u(t) = Au(t) + f(t)$ as a variation of constants formula, we find an interesting connection between $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ and a discrete resolvent family $\{S^n_{\alpha,\alpha}\}_{n\in\mathbb{N}}$ and then, by using the properties of $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$, we study the existence of solutions to the discrete subdiffusion equation ${}_C\nabla^\alpha u^n = Au^n + f^n, n \in \mathbb{N}$, where, based on the backward Euler method for a $\tau > 0$ given, ${}_C\nabla^\alpha u^n$ is an approximation of $\partial_t^\alpha u(t)$ at time $t_n := \tau n$. We study simultaneously the fractional derivative in the Caputo and Riemann-Liouville sense. We also provide error estimates and some experiments to illustrate the results.

1. Introduction

Let A be a closed linear operator defined in a Banach space X. The well known theory of C_0-semigroups of linear operators plays a fundamental role in the existence of solutions to the abstract differential equation of first-order $u'(t) = Au(t) + f(t)$, because its solution is given in terms of the variation of constants formula

$$u(t) = T(t)u(0) + \int_0^t T(t-s)f(s)ds,$$

where $\{T(t)\}_{t\geq 0}$ is the C_0-semigroup generated by A, see for instance [17]. The notion of C_0-semigroup can be seen as a particular case of a more general concept: the *resolvent families of operators.* This concept was introduced by Da Prato and Ianelli in [15, Definition 1] as an extension of the notion of C_0-semigroups of operators to study the existence of mild solutions to the integro-differential equation $u'(t) = \int_0^t k(t-s)Au(s)ds + f(t)$, for $t \geq 0$, under the initial condition $u(0) = u_0 \in X$, where $k \in L^1_{\text{loc}}(\mathbb{R}_+)$, $f \in C([0,T], X)$ and A is a closed linear operator which generates a resolvent family $\{U(t)\}_{t\geq 0}$. The solution to this integro-differential equation is given again in terms of its resolvent family by

$$u(t) = U(t)u_0 + \int_0^t U(t-s)f(s)ds.$$

These examples of families of operators show that the solution to certain abstract equations can be written in terms of those families. For this reason, the general theory of resolvent families, which allows to study the existence of solutions (and their properties) to a wide class of abstract equations, including Volterra equations [22, 52], abstract second order equations [5, 56], fractional differential equations [33, 34], among other, has had a rapid developed during the last two decades.

Fractional subdiffusion equations appear in many problems in physics and biological sciences, such as anomalous diffusion, fractional generalization of the kinetic equation, random walks, fluid flow, rheology, electrical networks, control theory of dynamical systems, viscoelasticity, chemical physics, signal processing, among other, see for instance [4, 24, 31, 45].

Again, to study the existence of solutions to fractional differential equations, a crucial tool are the fractional resolvent families, see [7, 8, 10, 16, 33, 34, 35, 50, 51, 57] and the references therein. More

2010 *Mathematics Subject Classification.* Primary 34A08; Secondary 65J10, 65M22,65M60.

Key words and phrases. Fractional differential equations, difference equations, resolvent families, backward Euler method.

concretely, if $0<\alpha<1$ and if we consider the subdiffusion equations

$$\partial_t^\alpha u(t) = Au(t) + f(t), t \geq 0, \quad u(0) = u_0, \tag{1.1}$$

and

$$^R\partial_t^\alpha u(t) = Au(t) + f(t), t \geq 0, \quad (g_{1-\alpha} * u)(0) = u_0, \tag{1.2}$$

where ∂_t^α and $^R\partial_t^\alpha$ denote, respectively, the Caputo and the Riemann-Liouville fractional derivatives, f is a suitable function, g_β is the function defined by $g_\beta(t) = t^{\beta-1}/\Gamma(\beta)$ (here $\Gamma(\cdot)$ denotes the Gamma function and $\beta > 0$), A is a closed and linear operator defined on X, (typically A is the second order operator), and x belongs to X, then the solutions to Problems (1.1)–(1.2) can be written, respectively, in terms of a variation of constants formula as

$$u(t) = S_{\alpha,1}(t)u_0 + \int_0^t S_{\alpha,\alpha}(t-s)f(s)ds, \tag{1.3}$$

and

$$u(t) = S_{\alpha,\alpha}(t)u_0 + \int_0^t S_{\alpha,\alpha}(t-s)f(s)ds, \tag{1.4}$$

where, for $\alpha, \beta > 0$, $S_{\alpha,\beta}(t)$ is the fractional resolvent family defined by

$$S_{\alpha,\beta}(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}d\lambda, \quad t \geq 0.$$

Here, γ is a suitable complex path where the resolvent operator $(\lambda^\alpha - A)^{-1}$ is well defined. The function $S_{\alpha,\beta}(t)$ corresponds to a generalization of the scalar Mittag-Leffler function, introduced by Mittag-Leffler and Wiman [18, 58, 59], which is defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{1}{2\pi i}\int_{Ha} e^\mu \mu^{\alpha-\beta}(\mu^\alpha - z)^{-1}d\mu, \quad \alpha, \beta > 0, z \in \mathbb{C}, \tag{1.5}$$

where, Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ counterclockwise.

There are important connections between resolvent families and the existence of solutions to fractional differential equations, see for instance [11, 12, 13, 32, 38, 46, 47, 48, 53] and the references therein. More specifically, in [11, 12, 13] the authors study the behavior of the resolvent family associated to the fractional differential equation $\partial_t^\alpha u(t) = Au(t) + f(t)$, where $1 < \alpha < 2$, A is sectorial operator and f is a suitable function, and then, the authors obtain the asymptotic behavior of a time discretization of this equation based on the backward Euler method for $\tau > 0$. On the other hand, in [46, 47, 48, 53] the authors study discretizations in time of integro-differential equations in Banach spaces, which can be seen as an integral version of some fractional differential equations. See also the Monograph [9] for a general study of discretizations of integro-differential equations. More recently, in [38] the author shows the existence of solutions to the Caputo fractional difference equation

$${}_C\Delta^\alpha u^n = Au^{n+1}, \quad n \in \mathbb{N}, \tag{1.6}$$

with the initial condition $u^0 = u_0 \in D(A)$, where $0 < \alpha < 1$, the operator ${}_C\Delta^\alpha u^n$ is an approximation of the Caputo fractional derivative $\partial_t^\alpha u(t)$ (at time $t = n$) which is defined by

$${}_C\Delta^\alpha u^n := \sum_{j=0}^n \frac{\Gamma(1-\alpha+n-j)}{\Gamma(1-\alpha)\Gamma(n-j+1)}(u^{j+1} - u^j),$$

where $u^j := \frac{1}{j!}\int_0^\infty e^{-t}t^j u(t)dt$. The solution to (1.6) is given by $u^n = S^n_{\alpha,1}(I-A)u_0$, where $\{S_{\alpha,1}(t)\}_{t\geq 0}$ is the resolvent family given in (1.3) and $S^n_{\alpha,1} = \frac{1}{n!}\int_0^\infty e^{-t}t^n S_{\alpha,1}(t)dt$, for all $n \in \mathbb{N}$.

The study of existence of solutions to fractional difference equations of Caputo and Riemann-Liouville type has been studied widely in the last years, see for instance the interesting papers [3, 6, 20] and the references given there. However, these discretizations for the Caputo and Riemann-Liouville fractional derivatives lack the time step $\tau > 0$. On the other hand, the authors in [27] and [39] study l_p-discrete maximal regularity of fractional evolution equations for the Caputo and Riemann-Liouville fractional derivatives on Banach space with the UMD property. See also [25, 26, 28, 29, 40] for different schemes of approximation of fractional models.

In this work, we give a discretization in time to equations (1.1) and (1.2) based on the backward Euler convolution method for $\tau > 0$ (see for instance [43, 44]). Here, by assuming that A is generator of a resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$, we study a strong connection between this resolvent family and the existence of solutions to the discrete equations for the Caputo and Riemann-Liouville derivative

$$(1.7) \qquad {}_C\nabla^\alpha u^n = Au^n + f^n, \quad n \in \mathbb{N} \quad \text{and} \quad ({}^R\nabla^\alpha u)^n = Au^n + f^n, \quad n \in \mathbb{N}.$$

Moreover, we show that the solution to these discrete fractional difference equations can be written as variation of constants formula, analogously to (1.3) and (1.4), but in terms of a discrete fractional resolvent family $\{S^n_{\alpha,\alpha}\}_{n\in\mathbb{N}}$.

The paper is organized as follows. In Section 2 we give the Preliminaries on resolvent families and continuous and discrete fractional calculus. Here, given a time step size $\tau > 0$, we present new discretizations ${}_C\nabla^\alpha u^n$ and ${}^R\nabla^\alpha u^n$ to $\partial_t^\alpha u$ and ${}^R\partial_t^\alpha u$, respectively. Moreover, we study the main properties of ${}_C\nabla^\alpha u^n$ and ${}^R\nabla^\alpha u^n$, in Theorem 2.7 we find an interesting connection between ${}_C\nabla^\alpha u^n$ and $\partial_t^\alpha u$ and we study the connection between the continuous and the discrete resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ and $\{S^n_{\alpha,\alpha}\}_{n\in\mathbb{N}}$, respectively. The Section 3 is devoted to the numerical scheme for (1.3) and (1.4). Here, by assuming that A is a sectorial operator we study the existence of solutions to (1.7). Moreover, we show that if A is the generator of a resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$, then the fractional difference equations (1.7) have a solution given in terms of the discrete resolvent family $\{S^n_{\alpha,\alpha}\}_{n\in\mathbb{N}}$. In Section 4 we study error estimates of the continuous and discrete solution and in Section 5 we provide some numerical experiments to illustrate the theoretical results.

2. Fractional resolvent families and continuous and discrete fractional calculus

2.1. **Resolvent families.** Let $X \equiv (X, \|\cdot\|)$ be a Banach space. The Banach space of all bounded and linear operators from X into X is denoted by $\mathcal{B}(X)$. If A is a closed linear operator on X, its resolvent set is denoted by $\rho(A)$, and the resolvent operator is defined by $R(\lambda, A) = (\lambda - A)^{-1}$ for all $\lambda \in \rho(A)$. The spectrum of A is defined by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. A family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is *exponentially bounded* if there exist real numbers $M > 0$ and $\omega \in \mathbb{R}$ such that

$$(2.8) \qquad \|S(t)\| \leq Me^{\omega t}, \quad t \geq 0.$$

We notice that if $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is exponentially bounded, then the Laplace transform of $S(t)$

$$\hat{S}(\lambda)x := \int_0^\infty e^{-\lambda t} S(t)x dt$$

exists for all $\mathrm{Re}\lambda > \omega$. If $\omega = 0$, then $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is bounded for all $t \geq 0$.

Definition 2.1. [2] *Let $\alpha, \beta > 0$ be given. Let A be a closed linear operator with domain $D(A)$ defined in a Banach space X. The operator A is called the generator of an (α,β)*-resolvent family *if there exist $\omega \geq 0$ and a strongly continuous function $S_{\alpha,\beta} : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\lambda^\alpha : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$ and*

$$\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_{\alpha,\beta}(t)x dt,$$

for all $\mathrm{Re}\lambda > \omega$ and $x \in X$. The family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is also called the (α,β)-resolvent family generated by A.

If we compare Definition 2.1 with the concept of (a, k)-regularized families introduced in [36] we observe that the function $t \mapsto S_{\alpha,\beta}(t)$, for $t \geq 0$, is a (g_α, g_β)-regularized family. Moreover, the function $S_{\alpha,\beta}(t)$ satisfies the following functional equation (see [2, 34, 41]):

$$S_{\alpha,\beta}(s)(g_\alpha * S_{\alpha,\beta})(t) - (g_\alpha * S_{\alpha,\beta})(s)S_{\alpha,\beta}(t) = g_\beta(s)(g_\alpha * S_{\alpha,\beta})(t) - g_\beta(t)(g_\alpha * S_{\alpha,\beta})(s),$$

for all $t, s \geq 0$. Moreover, a closed operator A generates a unique (α,β)-resolvent family. We recall that for $\mu > 0$, g_μ defines the function $g_\mu(t) := \frac{t^{\mu-1}}{\Gamma(\mu)}$ for all $t \geq 0$. We notice that g_β behaves like a delta function in the sense that $g_\beta * f \to f$ as $\beta \to 0$ and therefore, we define naturally $(g_0 * f)(t) := f(t)$ for all $t \geq 0$.

Moreover, if an operator A with domain $D(A)$ is the infinitesimal generator of a resolvent family $S_{\alpha,\beta}(t)$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t\to 0^+} \frac{S_{\alpha,\beta}(t)x - g_\beta(t)x}{g_{\alpha+\beta}(t)}.$$

For example, if $\alpha = \beta = 1$, then $S_{1,1}(t)$ corresponds to a C_0-semigroup, if $\alpha = 2, \beta = 1$, then $S_{2,1}(t)$ is a cosine family, and if $\alpha = \beta = 2$, then $S_{2,2}(t)$ is a sine family. See [5] for further details. If $\alpha > 0$ and $\beta = 1$, then $S_{\alpha,1}(t)$ is the solution operator introduced in [7, Definition 2.3]. The following result gives some properties of $S_{\alpha,\beta}(t)$. Its proof can be deduced from [36, Lemma 2.2 and Proposition 2.5] and the details are in [10, Proposition 3.10] and [1].

Proposition 2.2. *Let $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ be the (α,β)-resolvent family generated by A. Then,*

(1) $\displaystyle\lim_{t\to 0^+} \frac{S_{\alpha,\beta}(t)x}{g_\beta(t)} = x$, *for all $x \in X$.*
(2) *For all $x \in D(A)$ and $t \geq 0$ we have $S_{\alpha,\beta}(t)x \in D(A)$ and $AS_{\alpha,\beta}(t)x = S_{\alpha,\beta}(t)Ax$.*
(3) *For $x \in X$ and $t \geq 0$ we have $\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds \in D(A)$ and*

$$S_{\alpha,\beta}(t)x = g_\beta(t)x + A\int_0^t g_\alpha(t-s)S_{\alpha,\beta}(s)xds. \tag{2.9}$$

We say that an operator $A : D(A) \subset X \to X$ is said to be *sectorial of angle θ* if there are constants $\omega \in \mathbb{R}$, $M > 0$ and $\theta \in (\pi/2, \pi)$ such that $\rho(A) \supset S_{\theta,\omega} := \{z \in \mathbb{C} : z \neq \omega : |\arg(z-\omega)| < \theta\}$ and

$$\|(z-A)^{-1}\| \leq \frac{M}{|z-\omega|} \quad \text{for all} \quad z \in S_{\theta,\omega}.$$

In order to simplify the presentation of the results, we may assume, without lost of generality, that $\omega = 0$. If not so we can take the operator $A - \omega I$, which is also sectorial (here I denotes the identity operator in X). In that case, we write $A \in \mathrm{Sect}(\theta, M)$ and we denote the sector $S_{\theta,0}$ as S_θ. For further details on sectorial operators we refer to the reader to [17, 23].

Let A be a linear and closed operator whose resolvent set contains the negative half-line $(-\infty, 0]$, (for example, a sectorial operator with $\omega \geq 0$.) Given $0 \leq \varepsilon \leq 1$, X^ε denotes the domain of the fractional power A^ε, that is $X^\varepsilon := D(A^\varepsilon)$ endowed with the graph norm $\|x\|_\varepsilon = \|A^\varepsilon x\|$ (see for instance [42]). We notice that X^1 corresponds to the domain of A, and X^0 to the space X. It is a well known fact that if $0 < \varepsilon < 1$, and $x \in D(A)$, then there exists a constant $K \equiv K_\varepsilon > 0$ such that (see [42])

$$\|A^\varepsilon x\| \leq K\|Ax\|^\varepsilon \|x\|^{1-\varepsilon}. \tag{2.10}$$

2.2. **Continuous and discrete fractional calculus.** Now, we review some preliminaries on fractional calculus. For $\alpha > 0$, let $m = \lceil \alpha \rceil$ be the smallest integer m greater than or equal to α. The *Caputo fractional derivative of order* α of a C^m-differentiable function $f : \mathbb{R}_+ \to X$ is defined by

$$\partial_t^\alpha f(t) := \int_0^t g_{m-\alpha}(t-s)f^{(m)}(s)ds.$$

Similarly, the *Riemann-Liouville fractional derivative of order* α of $f : \mathbb{R}_+ \to X$ is defined by

$${}^R\partial_t^\alpha f(t) := \frac{d^m}{dt^m}\int_0^t g_{m-\alpha}(t-s)f(s)ds.$$

We observe that if $\alpha = m \in \mathbb{N}$, then $\partial_t^m f$ and ${}^R\partial_t^m f$ coincide with the usual derivative $\frac{d^m f}{dt^m}$ of order m. Moreover, if $0 < \alpha < 1$, then $\partial_t^{\alpha+1} f(t) = \partial_t^\alpha(\partial_t^1 f)(t) \neq \partial_t^1(\partial_t^\alpha f)(t)$ and ${}^R\partial_t^{\alpha+1} f(t) = \partial_t^1({}^R\partial_t^\alpha f)(t) \neq {}^R\partial_t^\alpha(\partial_t^1 f)(t)$, unless $f(0) = 0$. For further details on fractional calculus, we refer to the reader to [49].

The set of non-negative integer numbers is denoted by $\mathbb{N}_0$ and the non-negative real numbers by $\mathbb{R}_0^+$. For $\tau > 0$ fixed and $n \in \mathbb{N}_0$, we set

$$\rho_n^\tau(t) := e^{-\frac{t}{\tau}} \left(\frac{t}{\tau}\right)^n \frac{1}{\tau n!}.$$

We notice that $\rho_n^\tau(t) \geq 0$ for all $t \geq 0$ and $n \in \mathbb{N}_0$, and $\rho_n^\tau(t) = \tau^{-1}\rho_n(t/\tau)$ where $\rho_n(t) := e^{-t}t^n/n!$. Moreover, an easy computation shows that

$$\int_0^\infty \rho_n^\tau(t)dt = 1, \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Let $u : \mathbb{R}_0^+ \to X$ be a bounded and locally integrable function. We define the sequence $(u^n)_n$ by

$$u^n := \int_0^\infty \rho_n^\tau(t)u(t)dt, \quad n \in \mathbb{N}_0. \tag{2.11}$$

We notice that for small $\tau > 0$, the function ρ_n^τ behaves like a delta function at $t_n := n\tau$ and therefore, u^n is an approximation of $u(t_n)$.

For the Banach space X, $\mathcal{F}(\mathbb{R}_0^+; X)$ denotes the vectorial space consisting of all vector-valued functions $v : \mathbb{R}_0^+ \to X$. For $n \in \mathbb{N}$ we define v^n as in (2.11). The *backward Euler operator* $\nabla_\tau : \mathcal{F}(\mathbb{R}_0^+; X) \to \mathcal{F}(\mathbb{R}_0^+; X)$ is defined by

$$\nabla_\tau v^n := \frac{v^n - v^{n-1}}{\tau}, \quad n \in \mathbb{N}.$$

For $m \geq 2$, we define recursively $\nabla_\tau^m : \mathcal{F}(\mathbb{R}_0^+; X) \to \mathcal{F}(\mathbb{R}_0^+; X)$ as

$$\nabla_\tau^m v^n := \begin{cases} \nabla_\tau^{m-1}(\nabla_\tau v)^n, & n \geq m \\ 0, & n < m, \end{cases} \tag{2.12}$$

where $\nabla_\tau^1 \equiv \nabla_\tau$ and ∇_τ^0 is the identity operator. The operator ∇_τ^m is called the *backward difference operator of order* m. We notice that if $v \in \mathcal{F}(\mathbb{R}_0^+; X)$ then

$$(\nabla_\tau^m v)^n = \frac{1}{\tau^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j v^{n-j}, \quad n \in \mathbb{N}.$$

Now, we define the sequence

$$k_\tau^\alpha(n) := \tau \int_0^\infty \rho_n^\tau(t) g_\alpha(t)dt, \quad n \in \mathbb{N}_0, \alpha > 0. \tag{2.13}$$

An easy computation shows that

$$k_\tau^\alpha(n) = \frac{\tau^\alpha \Gamma(\alpha + n)}{\Gamma(\alpha)\Gamma(n+1)} = \tau \frac{\Gamma(\alpha + n)}{\Gamma(n+1)} g_\alpha(\tau), \quad n \in \mathbb{N}_0, \alpha > 0. \tag{2.14}$$

The next definitions were introduced in [37, Definitions 2.5, 2.7 and 2.8] in case $\tau = 1$. See also [21, Chapter 3] and the references therein. We give here the definition for all $\tau > 0$.

Definition 2.3. *Let $\alpha > 0$. The α^{th}−fractional sum of $v \in \mathcal{F}(\mathbb{R}; X)$ is defined by*

$$(\nabla_\tau^{-\alpha} v)^n := \sum_{j=0}^{n} k_\tau^\alpha(n-j)v^j, \quad n \in \mathbb{N}_0. \tag{2.15}$$

We now introduce the fractional difference operators in the sense of Caputo and Riemann-Liouville.

Definition 2.4. *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. The* Caputo fractional backward difference operator of order α, *${}_C\nabla^\alpha : \mathcal{F}(\mathbb{R}_+; X) \to \mathcal{F}(\mathbb{R}_+; X)$, is defined by*

$$({}_C\nabla^\alpha v)^n := \nabla_\tau^{-(m-\alpha)}(\nabla_\tau^m v)^n, \quad n \in \mathbb{N},$$

where $m - 1 < \alpha < m$.

Here, as in [21, Chapter 1, Section 1.5] we define by convention

$$\sum_{j=0}^{-k} v^j = 0,$$

for all $k \in \mathbb{N}$.

Definition 2.5. *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. The* Riemann-Liouville fractional backward difference operator of order α, ${}^R\nabla^\alpha : \mathcal{F}(\mathbb{R}_+; X) \to \mathcal{F}(\mathbb{R}_+; X)$, *is defined by*

$$({}^R\nabla^\alpha v)^n := \nabla_\tau^m (\nabla_\tau^{-(m-\alpha)} v)^n, \quad n \in \mathbb{N},$$

where $m - 1 < \alpha < m$.

In both definitions, if $\alpha \in \mathbb{N}_0$, then the fractional backward difference operators ${}_C\nabla^\alpha$ and ${}^R\nabla^\alpha$ are defined as the backward difference operator ∇_τ^α.

We notice that if $0 < \alpha < 1$, then

$${}_C\nabla^{\alpha+1} v^n = \nabla_\tau^{-(2-(\alpha+1))} (\nabla_\tau^2 v)^n = \nabla_\tau^{-(1-\alpha)} (\nabla_\tau (\nabla_\tau v))^n = {}_C\nabla^\alpha (\nabla_\tau^1 v)^n = {}_C\nabla^\alpha ({}_C\nabla^1 v)^n$$

and

$${}^R\nabla^{\alpha+1} v^n = \nabla_\tau^2 (\nabla_\tau^{-(2-(\alpha+1))} v)^n = \nabla_\tau^1 (\nabla_\tau^1 (\nabla_\tau^{-(1-\alpha)} v))^n = \nabla_\tau^1 ({}^R\nabla^\alpha v)^n = {}^R\nabla^1 ({}^R\nabla^\alpha v)^n,$$

for all $n \in \mathbb{N}$.

However, ${}_C\nabla^{\alpha+1} v^n \neq {}_C\nabla^1 ({}_C\nabla^\alpha v)^n$ and ${}^R\nabla^{\alpha+1} v^n \neq {}^R\nabla^\alpha ({}^R\nabla^1 v)^n$. In fact, if $v(t) = tv_0$, where $0 \neq v_0 \in X$, then $\nabla_\tau^1 v^n = v_0$ for all $n \geq 1$ and $\nabla_\tau^2 v^n = 0$ for all $n \geq 2$. This implies that for $n \geq 2$ we obtain

$${}_C\nabla^{\alpha+1} v^n = \nabla_\tau^{-(1-\alpha)} (\nabla_\tau^2 v)^n = \sum_{j=2}^{n} k_\tau^{1-\alpha}(n-j)(\nabla_\tau^2 v)^j = 0.$$

On the other hand,

$$\begin{aligned} {}_C\nabla^1 ({}_C\nabla^\alpha v)^n &= \frac{1}{\tau} \left({}_C\nabla_\tau^\alpha v^n - {}_C\nabla_\tau^\alpha v^{n-1} \right) \\ &= \frac{1}{\tau} \left(\sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j)(\nabla_\tau^1 v)^j - \sum_{j=1}^{n-1} k_\tau^{1-\alpha}(n-1-j)(\nabla_\tau^1 v)^j \right) \\ &= \frac{1}{\tau} \left(\sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j) - \sum_{j=1}^{n-1} k_\tau^{1-\alpha}(n-1-j) \right) v_0 \\ &= \frac{1}{\tau} k_\tau^{1-\alpha}(n-1) v_0, \end{aligned}$$

for all $n \geq 2$. Since $k_\tau^{1-\alpha}(1) = (1-\alpha)\tau^{1-\alpha}$, we conclude that if $\alpha > 0$ and $n = 2$, then

$${}_C\nabla^1 ({}_C\nabla^\alpha v)^2 = (1-\alpha)\tau^{-\alpha} v_0.$$

Therefore, ${}_C\nabla^{\alpha+1} v^n \neq {}_C\nabla^1 ({}_C\nabla^\alpha v)^n$.

Now, if v is the constant vector-valued function $v(t) = v_0 \in X$, then $\nabla_\tau^1 v^n = 0$ for all $n \in \mathbb{N}$ and thus

$${}^R\nabla^\alpha (\nabla^1 v)^n = \nabla_\tau^1 (\nabla_\tau^{-(1-\alpha)} \nabla_\tau v)^n = \frac{1}{\tau} \left(\nabla_\tau^{-(1-\alpha)} (\nabla_\tau v)^n - \nabla_\tau^{-(1-\alpha)} (\nabla_\tau v)^{n-1} \right).$$

Since

$$\nabla_\tau^{-(1-\alpha)} (\nabla_\tau v)^m = \sum_{j=0}^{m} k_\tau^{1-\alpha}(m-j)(\nabla_\tau^1 v)^j = 0,$$

we obtain ${}^R\nabla^\alpha(\nabla^1 v)^n = 0$ for all $n \geq 2$. On the other hand,

$$\begin{aligned} {}^R\nabla^{\alpha+1}v^n &= \nabla^2_\tau(\nabla_\tau^{-(2-(\alpha+1))}v)^n \\ &= \frac{1}{\tau^2}\left(\nabla_\tau^{-(1-\alpha)}v^n - 2\nabla_\tau^{-(1-\alpha)}v^{n-1} + \nabla_\tau^{-(1-\alpha)}v^{n-2}\right). \end{aligned}$$

Since

$$\nabla_\tau^{-(1-\alpha)}v^m = \sum_{j=0}^m k_\tau^{1-\alpha}(m-j)v^j = \sum_{j=0}^m k_\tau^{1-\alpha}(m-j)v_0,$$

and

$$k_\tau^{1-\alpha}(0) = \tau^{1-\alpha}, \quad k_\tau^{1-\alpha}(1) = (1-\alpha)\tau^{1-\alpha} \quad \text{and} \quad k_\tau^{1-\alpha}(2) = \frac{(1-\alpha)(2-\alpha)}{2}\tau^{1-\alpha},$$

we obtain for $n = 2$ and $\alpha > 0$ that

$$({}^R\nabla^{\alpha+1}v)^2 = \frac{1}{\tau^2}(k_\tau^{1-\alpha}(2) - k_\tau^{1-\alpha}(1))v_0 = -\frac{\alpha(1-\alpha)}{2\tau^{1+\alpha}}v_0.$$

Therefore, ${}^R\nabla^{\alpha+1}v^n \neq {}^R\nabla^\alpha({}^R\nabla^1 v)^n$. However, there is an interesting connection between the Riemann-Liouville and Caputo fractional difference operators: if $v \in \mathcal{F}(\mathbb{R}_+; X)$ and $0 < \alpha < 1$, then

$$\begin{aligned} {}^R\nabla^\alpha(\nabla^1 v)^n = \nabla^1_\tau(\nabla_\tau^{-(1-\alpha)}(\nabla^1 v))^n &= \frac{1}{\tau}\left(\nabla_\tau^{-(1-\alpha)}(\nabla^1 v)^n - \nabla_\tau^{-(1-\alpha)}(\nabla^1 v)^{n-1}\right) \\ &= \frac{1}{\tau}\left({}_C\nabla^\alpha v^n - {}_C\nabla^\alpha v^{n-1}\right) \\ &= \nabla^1({}_C\nabla^\alpha v)^n, \end{aligned}$$

for all $n \in \mathbb{N}$. We summarize the above properties in the following Proposition.

Proposition 2.6. *If $0 < \alpha < 1$ and $n \in \mathbb{N}$, then*

(1) ${}_C\nabla^{\alpha+1}v^n = {}_C\nabla^\alpha(\nabla^1 v)^n$,
(2) ${}^R\nabla^{\alpha+1}v^n = \nabla^1({}^R\nabla^\alpha v)^n$, *and*
(3) ${}^R\nabla^\alpha(\nabla^1 v)^n = \nabla^1({}_C\nabla^\alpha v)^n$.

The next result relates the Caputo (respectively, the Riemann-Liouville) fractional derivative and the Caputo (respectively, the Riemann-Liouville) difference operator. For the Riemann-Liouville fractional derivative, the case $\tau = 1$ can be found in [38].

Theorem 2.7. *Let $0 < \alpha < 1$.*

(1) *If $u : [0,\infty) \to X$ is differentiable and bounded, then*

$$\int_0^\infty \rho_n^\tau(t)\partial_t^\alpha u(t)dt = {}_C\nabla^\alpha u^n, \tag{2.16}$$

for all $n \in \mathbb{N}$.

(2) *If $u : [0,\infty) \to X$ is locally integrable and bounded, then*

$$\int_0^\infty \rho_n^\tau(t)\,{}^R\partial_t^\alpha u(t)dt = {}^R\nabla^\alpha u^n, \tag{2.17}$$

for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. The Fubini's theorem implies

$$\begin{aligned}\int_0^\infty \rho_n^\tau(t)\partial_t^\alpha u(t)dt &= \int_0^\infty \rho_n^\tau(t)(g_{1-\alpha} * u')(t)dt \\ &= \int_0^\infty \rho_n^\tau(t)\int_0^t g_{1-\alpha}(t-s)u'(s)dsdt \\ &= \int_0^\infty u'(s)\int_s^\infty \rho_n^\tau(t)g_{1-\alpha}(t-s)dtds \\ &= \int_0^\infty u'(s)\int_0^\infty e^{-\frac{1}{\tau}(r+s)}\frac{(r+s)^n}{\tau^n}\frac{1}{\tau n!}g_{1-\alpha}(r)drds \\ &= \int_0^\infty u'(s)e^{-\frac{s}{\tau}}\int_0^\infty e^{-\frac{r}{\tau}}\sum_{j=0}^n \binom{n}{j} r^{n-j}s^j\frac{1}{\tau^n}\frac{1}{\tau n!}g_{1-\alpha}(r)drds \\ &= \sum_{j=0}^n \int_0^\infty u'(s)e^{-\frac{s}{\tau}}\left(\frac{s}{\tau}\right)^j\frac{1}{\tau j!}\tau\int_0^\infty e^{-\frac{r}{\tau}}\left(\frac{r}{\tau}\right)^{n-j}\frac{1}{\tau(n-j)!}g_{1-\alpha}(r)drds \\ &= \sum_{j=0}^n \int_0^\infty \rho_j^\tau(s)u'(s)ds\tau\int_0^\infty \rho_{n-j}^\tau(r)g_{1-\alpha}(r)dr \\ &= \sum_{j=0}^n k_\tau^{1-\alpha}(n-j)\int_0^\infty \rho_j^\tau(s)u'(s)ds.\end{aligned}$$

Since u is a bounded function and

$$\frac{d\rho_k^\tau(t)}{dt} = \frac{1}{\tau}\left(\rho_{k-1}^\tau(t) - \rho_k^\tau(t)\right) \tag{2.18}$$

for all $k \geq 1$, we obtain by integration by parts that

$$\int_0^\infty \rho_j^\tau(s)u'(s)ds = \frac{1}{\tau}\left(\int_0^\infty \rho_j^\tau(s)u(s)ds - \int_0^\infty \rho_{j-1}^\tau(s)u(s)ds\right) = \frac{1}{\tau}\left(u^j - u^{j-1}\right) = (\nabla_\tau^1 v)^j,$$

which implies that

$$\int_0^\infty \rho_n^\tau(t)\partial_t^\alpha u(t)dt = \sum_{j=0}^n k_\tau^{1-\alpha}(n-j)(\nabla_\tau^1 v)^j = \nabla_\tau^{-(1-\alpha)}(\nabla_\tau^1 u)(n) = {}_C\nabla^\alpha u^n,$$

and the proof of (2.16) is finished. The proof of (2.17) follows analogously. □

Now, for a family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$, we define the sequence

$$S^n x := \int_0^\infty \rho_n^\tau(t)S(t)xdt, \quad n \in \mathbb{N}_0, x \in X.$$

If $c : \mathbb{R}_+ \to \mathbb{C}$ is a continuous and bounded function

$$c^n := \int_0^\infty \rho_n^\tau(t)c(t)dt, \quad n \in \mathbb{N}_0,$$

and we define the discrete convolution

$$(c \star S)^n := \sum_{k=0}^n c^{n-k}S^k, \quad n \in \mathbb{N}_0.$$

Finally, we recall that for a vector-valued function $f : \mathbb{R}_+ \to X$, the sequence f^n is defined as

$$f^n := \int_0^\infty \rho_n^\tau(t)f(t)dt, \quad n \in \mathbb{N}_0.$$

The proof of the following result is given in [38] for $\tau = 1$.

Theorem 2.8. *Let $c : \mathbb{R}_+ \to \mathbb{C}$ be Laplace transformable such that $\hat{c}(1/\tau)$ exists, and let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1/\tau)$ exists. Then, for all $x \in X$,*

$$\int_0^\infty \rho_n^\tau(t)(c * S)(t)x dt = \tau(c \star S)^n x, \quad n \in \mathbb{N}_0.$$

Proof. If $g : \mathbb{R}_0^+ \to X$ is a bounded and locally integrable function, then

$$\int_0^\infty \rho_n^\tau(t)g(t)dt = \tau^{-1}\int_0^\infty e^{-\frac{t}{\tau}}\left(\frac{t}{\tau}\right)^n \frac{1}{n!}g(t)dt = \frac{\tau^{-1}}{\tau^n}\int_0^\infty e^{-\frac{t}{\tau}}\frac{t^n}{n!}g(t)dt = \frac{\tau^{-1}}{\tau^n}\frac{(-1)^n}{n!}[\hat{g}(\lambda)]^{(n)} \mid_{\lambda=\frac{1}{\tau}},$$

which implies

$$\begin{aligned}\int_0^\infty \rho_n^\tau(t)(c * S)(t)x dt &= \frac{\tau^{-1}}{\tau^n}\frac{(-1)^n}{n!}[\widehat{(c * S)}(\lambda)]^{(n)}x \mid_{\lambda=\frac{1}{\tau}} \\ &= \frac{\tau^{-1}}{\tau^n}\frac{(-1)^n}{n!}[\hat{c}(\lambda)\hat{S}(\lambda)]^{(n)}x \mid_{\lambda=\frac{1}{\tau}} \\ &= \frac{\tau^{-1}}{\tau^n}\frac{(-1)^n}{n!}\sum_{k=0}^n \binom{n}{k}[c(\lambda)]^{(n-k)}[\hat{S}(\lambda)]^{(k)}x \mid_{\lambda=\frac{1}{\tau}}.\end{aligned}$$

Since

$$[\hat{S}(\lambda)]^{(k)}x \mid_{\lambda=\frac{1}{\tau}} = \frac{d^k\hat{S}(\lambda)}{d\lambda^k}x \mid_{\lambda=\frac{1}{\tau}} = (-1)^k\int_0^\infty e^{-\frac{1}{\tau}t}t^k S(t)x dt = k!(-1)^k\int_0^\infty e^{-\frac{1}{\tau}t}\frac{t^k}{k!}S(t)x dt,$$

we obtain

$$[\hat{S}(\lambda)]^{(k)}x \mid_{\lambda=\frac{1}{\tau}} = k!(-1)^k\tau^{k+1}\int_0^\infty \rho_k^\tau(t)S(t)x dt = k!(-1)^k\tau^{k+1}S^k x,$$

for all $k \in \mathbb{N}_0$. Similarly, $[\hat{c}(\lambda)]^{(k)} = k!(-1)^k\tau^{k+1}c^k$, for all $k \in \mathbb{N}_0$. Therefore,

$$\int_0^\infty \rho_n^\tau(t)(c*S)(t)x dt = \frac{\tau^{-1}}{\tau^n}\frac{(-1)^n}{n!}\sum_{k=0}^n\binom{n}{k}(-1)^{n-k}(n-k)!\tau^{n-k}\tau c^{n-k}(-1)^k k!\tau^k\tau S^k x = \tau\sum_{k=0}^n c^{n-k}S^k x.$$

□

Since $\hat{g}_\alpha(\lambda) = \frac{1}{\lambda^\alpha}$, for all $\mathrm{Re}\lambda > 0$, we obtain

$$g_\alpha^n = \int_0^\infty \rho_n^\tau(t)g_\alpha(t)dt = \frac{1}{\tau}k_\tau^\alpha(n)$$

for all $n \in \mathbb{N}$, and by Theorem 2.8, we have the following Corollary.

Corollary 2.9. *Let $\alpha > 0$. Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1/\tau)$ exists. Then for all $x \in X$,*

$$\int_0^\infty \rho_n^\tau(t)(g_\alpha * S)(t)x dt = \sum_{j=0}^n k_\tau^\alpha(n-j)S^j x, \quad n \in \mathbb{N}_0.$$

Moreover, since $(g_\alpha * g_\beta)(t) = g_{\alpha+\beta}(t)$ for all $\alpha, \beta > 0$, we obtain by (2.13),(2.14) and Corollary 2.9

$$k_\tau^{\alpha+\beta}(n) = \tau\int_0^\infty \rho_n^\tau(t)g_{\alpha+\beta}(t)dt = \tau\int_0^\infty \rho_n^\tau(t)(g_\alpha * g_\beta)(t)dt = \tau\sum_{j=0}^n k_\tau^\alpha(n-j)g_\beta^j = \sum_{j=0}^n k_\tau^\alpha(n-j)k_\tau^\beta(j),$$

which implies

$$k_\tau^{\alpha+\beta}(n) = (g_\alpha \star g_\beta)^n = \sum_{j=0}^n k_\tau^\alpha(n-j)k_\tau^\beta(j), \tag{2.19}$$

for all $n \in \mathbb{N}_0$.

The proof of the next result follows similarly to the proof of Theorem 2.8. We omit the details.

Proposition 2.10. *Let $f : \mathbb{R}_+ \to X$ be Laplace transformable such that $\hat{f}(1/\tau)$ exists, and let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be strongly continuous and Laplace transformable such that $\hat{S}(1/\tau)$ exists. Then,*

$$\int_0^\infty \rho_n^\tau(t)(S * f)(t)xdt = \tau(S \star f)^n x = \tau \sum_{j=0}^n S^{n-j} f^j, \quad n \in \mathbb{N}_0.$$

3. Numerical scheme

Let $\alpha, \beta > 0$. For $n \geq 1$, we define the bounded operators $D^n_{\alpha,\beta} : X \to X$ by

$$D^n_{\alpha,\beta} := \frac{1}{2\pi i} \int_\gamma r(\tau\lambda)^n \lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1} d\lambda, \tag{3.20}$$

where $r(z) := \frac{1}{1-z}$ and γ denotes a suitable path that connects $-i\infty$ and $+i\infty$ with increasing imaginary part. If $f_{\alpha,\beta}(z) := \tau^{\beta-1}(1-z)^{\alpha-\beta}\left((1-z)^\alpha - \tau^\alpha A\right)^{-1}$, then $f_{\alpha,\beta}$ is a holomorphic operator-valued mapping on $|z| = r < 1$, and therefore, Cauchy's formula implies that

$$f_{\alpha,\beta}(z) = \sum_{n=0}^\infty a_n^{\alpha,\beta} z^n, \tag{3.21}$$

where

$$a_n^{\alpha,\beta} = \frac{1}{2\pi i} \int_{|z|=r} \tau^{\beta-1} \frac{(1-z)^{\alpha-\beta}}{z^{n+1}} \left((1-z)^\alpha - \tau^\alpha A\right)^{-1} dz.$$

It is easy to see that

$$a_n^{\alpha,\beta} = D^{n+1}_{\alpha,\beta} \tag{3.22}$$

for all $n \geq 0$. Now, we define $F_{\alpha,\beta}(z)$ as the Laplace transform of $S_{\alpha,\beta}(t)$, that is,

$$F_{\alpha,\beta}(z) := \int_0^\infty e^{-zt} S_{\alpha,\beta}(t)dt$$

Let $L^n_{\alpha,\beta} : X \to X$ be the generating operators satisfying

$$\sum_{n=0}^\infty L^n_{\alpha,\beta} z^n = \frac{1}{\tau} F_{\alpha,\beta}\left(\frac{1-z}{\tau}\right).$$

An easy computation shows that

$$\frac{1}{\tau} F_{\alpha,\beta}\left(\frac{1-z}{\tau}\right) = f_{\alpha,\beta}(z),$$

and then, by (3.22), we conclude that

$$L^n_{\alpha,\beta} = D^{n+1}_{\alpha,\beta} \tag{3.23}$$

for all $n \in \mathbb{N}_0$. As a consequence, we have the following result.

Theorem 3.11. *Let $\alpha, \beta > 0$. Assume that A is the generator of an (α,β)-resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ resolvent family. Then*

$$D^{n+1}_{\alpha,\beta} x = \int_0^\infty \rho_n^\tau(t) S_{\alpha,\beta}(t)xdt, \tag{3.24}$$

for all $n \in \mathbb{N}$ and $x \in X$, and therefore

$$S^n_{\alpha,\beta} = D^{n+1}_{\alpha,\beta}.$$

Proof. Since

$$\sum_{n=0}^{\infty} L^n_{\alpha,\beta} z^n = \frac{1}{\tau} F_{\alpha,\beta}\left(\frac{1-z}{\tau}\right) = \frac{1}{\tau}\int_0^\infty e^{-\frac{(1-z)}{\tau}t} S_{\alpha,\beta}(t)dt$$

and

$$\frac{1}{\tau} e^{-\frac{(1-z)}{\tau}t} = \sum_{n=0}^{\infty} \rho_n^\tau(t) z^n,$$

we obtain

$$\sum_{n=0}^{\infty} L^n_{\alpha,\beta} z^n = \sum_{n=0}^{\infty}\int_0^\infty \rho_n^\tau(t) S_{\alpha,\beta}(t) dt z^n,$$

which implies that

$$L^n_{\alpha,\beta} = \int_0^\infty \rho_n^\tau(t) S_{\alpha,\beta}(t)dt,$$

for all $n \in \mathbb{N}_0$, and the result follows from (3.23). □

3.1. **Caputo fractional difference equations.** Now, for $0 < \alpha < 1$, we consider the equation (1.1):

$$\partial_t^\alpha u(t) = Au(t) + f(t), \quad t \geq 0,$$

with the initial condition $u(0) = u_0$. Multiplying this by $\rho_n^\tau(t)$ and integrating over $[0, \infty)$ we obtain by Theorem 2.7 the backward Euler scheme

$$_C\nabla^\alpha u^n = Au^n + f^n, \quad n \in \mathbb{N}, \tag{3.25}$$

for all $n \in \mathbb{N}$, with the initial condition $u^0 = u_0$. We first assume that A is a sectorial operator and $u^0 \in \ker(A)$, that is, $u^0 \in D(A)$ and $Au^0 = 0$. By Definition (2.12), $(\nabla^1_\tau u)^0 = 0$, which implies that $_C\nabla^\alpha u^0 = \nabla^{-(1-\alpha)}(\nabla^1 u)^0 = 0$. Moreover, if $f^0 = 0$, then we can consider the equation (3.25) for all $n \in \mathbb{N}_0$. By definition, we can write

$$_C\nabla^\alpha u^n = \nabla_\tau^{-(1-\alpha)}(\nabla_\tau u)^n = \sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)(\nabla^1_\tau u)^j = \sum_{j=1}^{n-1} k_\tau^{1-\alpha}(n-j)(\nabla^1_\tau u)^j + \tau^{-\alpha}(u^n - u^{n-1}),$$

for all $n \in \mathbb{N}$.

Thus, the scheme (3.25) is equivalent to

$$(\tau^{-\alpha} - A)u^n = \tau^{-\alpha}u^{n-1} - \sum_{j=1}^{n-1} k_\tau^{1-\alpha}(n-j)(\nabla^1_\tau u)^j + f^n, \tag{3.26}$$

for all $n \in \mathbb{N}$. We notice that this is an implicit scheme, and to obtain u^n from $u^{n-1}, ..., u^0$ we need to solve (3.26). To this end, since A is a sectorial operator, we can take τ small enough (for instance $\max\{0, \omega\}\tau^\alpha < 1$) in order to obtain that $(\tau^{-\alpha} - A)$ is invertible.

Now, we will write (3.25) in terms of generating functions. We set for $\beta > 0$ and $|z| < 1$

$$U(z) := \sum_{n=0}^{\infty} u^n z^n, \quad F(z) := \sum_{n=0}^{\infty} f^n z^n, \quad Q^\beta(z) := \sum_{n=0}^{\infty} k_\tau^\beta(n) z^n = \left(\frac{\tau}{1-z}\right)^\beta.$$

Multiplying (3.25) by z^n and summing up in $n \in \mathbb{N}_0$, we obtain

$$\sum_{n=0}^{\infty} {_C\nabla^\alpha} u^n z^n = AU(z) + F(z). \tag{3.27}$$

Since

$$\begin{aligned}\sum_{n=0}^{\infty}({}_C\nabla^{\alpha}u^n)z^n &= \sum_{n=0}^{\infty}\nabla_{\tau}^{-(1-\alpha)}(\nabla_{\tau}^{1}u)^n z^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}k_{\tau}^{1-\alpha}(n-j)(\nabla_{\tau}^{1}u)^j\right)z^n \\ &= \left(\sum_{n=0}^{\infty}k_{\tau}^{1-\alpha}(n)z^n\right)\left(\sum_{n=0}^{\infty}(\nabla_{\tau}^{1}u)^n z^n\right) \\ &= Q^{1-\alpha}(z)\sum_{n=0}^{\infty}(\nabla_{\tau}^{1}u)^n z^n.\end{aligned}$$

Now, since $(\nabla_{\tau}^{1}u)^0=0$ we obtain

$$\sum_{n=0}^{\infty}(\nabla_{\tau}^{1}u)^n z^n=\frac{1}{\tau}\sum_{n=1}^{\infty}(u^n-u^{n-1})z^n=\frac{1}{\tau}\left((1-z)U(z)-u^0\right).$$

Hence (3.27) reads

$$\left(\frac{(1-z)}{\tau}Q^{1-\alpha}(z)-A\right)U(z)=\frac{1}{\tau}Q^{1-\alpha}(z)u^0+F(z),$$

which is equivalent to

$$(I-Q^{\alpha}(z)A)U(z)=\frac{1}{1-z}u^0+Q^{\alpha}(z)F(z). \tag{3.28}$$

Since A is a sectorial operator, and $\mathrm{Re}((1-z)/\tau)>0$ for all $|z|=r<1$, we obtain that $\left(\frac{1-z}{\tau}\right)^{\alpha}=\frac{1}{Q^{\alpha}(z)}\in\rho(A)$, $(I-Q^{\alpha}(z)A)$ is invertible and we can write (3.28) as

$$U(z)=\frac{1}{1-z}(I-Q^{\alpha}(z)A)^{-1}u^0+Q^{\alpha}(z)(I-Q^{\alpha}(z)A)^{-1}F(z). \tag{3.29}$$

Noticing that $Q^{\alpha}(z)=\left(\frac{\tau}{1-z}\right)^{\alpha}=\tau^{\alpha}(1-z)^{-\alpha}$, we can write

$$\frac{1}{1-z}(I-Q^{\alpha}(z)A)^{-1}=(1-z)^{\alpha-1}\left((1-z)^{\alpha}-\tau^{\alpha}A\right)^{-1}=f_{\alpha,1}(z).$$

By (3.21) we obtain

$$f_{\alpha,1}(z)=\sum_{n=0}^{\infty}a_n^{\alpha,1}z^n,$$

where, for $n\in\mathbb{N}_0$,

$$a_n^{\alpha,1}=\frac{1}{2\pi i}\int_{|z|=r}\frac{(1-z)^{\alpha-1}}{z^{n+1}}\left((1-z)^{\alpha}-\tau^{\alpha}A\right)^{-1}dz=\frac{1}{2\pi i}\int_{|z|=r}\frac{1}{(1-z)z^{n+1}}(I-Q^{\alpha}(z)A)^{-1}dz.$$

The relation between $a_n^{\alpha,1}$ and $D_{\alpha,1}^{n}$ given in (3.22) gives $a_n^{\alpha,1}=D_{\alpha,1}^{n+1}$ for all $n\geq 0$ and thus,

$$\frac{1}{1-z}(I-Q^{\alpha}(z)A)^{-1}=f_{\alpha,1}(z)=\sum_{n=0}^{\infty}D_{\alpha,1}^{n+1}z^n. \tag{3.30}$$

On the other hand, since $(I-Q^{\alpha}(z)A)^{-1}$ is holomorphic on $|z|<1$, $Q^{\alpha}(z)(I-Q^{\alpha}(z)A)^{-1}$ is holomorphic as well on $|z|<1$. Since,

$$Q^{\alpha}(z)(I-Q^{\alpha}(z)A)^{-1}=\tau f_{\alpha,\alpha}(z)=\tau\sum_{n=0}^{\infty}a_n^{\alpha,\alpha}z^n=\tau\sum_{n=0}^{\infty}D_{\alpha,\alpha}^{n+1}z^n,$$

we obtain

$$Q^\alpha(z)(I-Q^\alpha(z)A)^{-1}F(z)=\tau\left(\sum_{n=0}^{\infty}D_{\alpha,\alpha}^{n+1}z^n\right)\left(\sum_{n=0}^{\infty}f^nz^n\right)=\tau\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}D_{\alpha,\alpha}^{n+1-j}f^j\right)z^n. \tag{3.31}$$

Replacing (3.30) and (3.31) in (3.29) we obtain

$$U(z)=\sum_{n=1}^{\infty}D_{\alpha,1}^{n+1}z^nu^0+\tau\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}D_{\alpha,\alpha}^{n+1-j}f^j\right)z^n$$

which implies that

$$u^n=D_{\alpha,1}^{n+1}u^0+\tau\sum_{j=0}^{n}D_{\alpha,\alpha}^{n+1-j}f^j,$$

for all $n\in\mathbb{N}$. We have proved the following result.

Theorem 3.12. *Let A be a sectorial operator in a Banach space X. If $u_0\in\ker(A)$ and $f^0=0$, then the fractional difference equation*

$$_C\nabla^\alpha u^n=Au^n+f^n,\quad n\in\mathbb{N}, \tag{3.32}$$

with the initial condition $u^0=u_0$, has a unique solution given by

$$u^n=D_{\alpha,1}^{n+1}u_0+\tau\sum_{j=0}^{n}D_{\alpha,\alpha}^{n+1-j}f^j, \tag{3.33}$$

for all $n\in\mathbb{N}$.

In the next result, we assume that u_0 merely belongs to X, and we show that even if u^0 is not in $D(A)$, we can obtain existence of solutions.

Theorem 3.13. *Let $\tau>0$. Let A be the generator of an (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ exponentially bounded with $\|S_{\alpha,\alpha}(t)\|\leq Me^{\omega t}$, where $\omega<\frac{1}{\tau}$. If $u^0\in X$ and f is bounded, then the fractional difference equation*

$$_C\nabla^\alpha u^n=Au^n+f^n,\quad n\in\mathbb{N},$$

with the initial condition $u^0=u_0$, has a unique solution given by

$$u^n=S_{\alpha,1}^nu_0+\tau(S_{\alpha,\alpha}\star f)^n, \tag{3.34}$$

for all $n\in\mathbb{N}$, where $S_{\alpha,1}(t)=(g_{1-\alpha}*S_{\alpha,\alpha})(t)$.

Proof. As in the proof of Theorem 2.8, we obtain $S_{\alpha,1}^nx\in D(A)$ for all $n\in\mathbb{N}_0$ and $x\in X$. By Proposition 2.2 we obtain

$$S_{\alpha,1}(t)x=x+A\int_0^t g_\alpha(t-s)S_{\alpha,1}(s)xds,$$

for all $t\geq 0$ and $x\in X$. Multiplying this equality by $\rho_j^\tau(t)$ and integrating over $[0,\infty)$ we have by Corollary 2.9 that for all $j\geq 0$,

$$S_{\alpha,1}^jx=\int_0^\infty\rho_j^\tau(t)S_{\alpha,1}(t)xdt=\int_0^\infty\rho_j^\tau(t)xdt+A\int_0^\infty\rho_j^\tau(t)(g_\alpha*S_{\alpha,1})(t)xdt=x+A\sum_{l=0}^{j}k_\tau^\alpha(j-l)S_{\alpha,1}^lx. \tag{3.35}$$

By definition, for all $n\in\mathbb{N}$, we have

$$_C\nabla^\alpha(S_{\alpha,1}x)^n=\nabla_\tau^{-(1-\alpha)}\nabla_\tau^1(S_{\alpha,1}x)^n=\sum_{j=0}^{n}k_\tau^{1-\alpha}(n-j)(\nabla_\tau^1S_{\alpha,1}x)^j,$$

and by (3.35) we get

$$(\nabla_\tau^1 S_{\alpha,1}x)^j = \frac{1}{\tau}(S_{\alpha,1}^j x - S_{\alpha,1}^{j-1}x) = \frac{A}{\tau}\sum_{l=0}^{j} k_\tau^\alpha(j-l)S_{\alpha,1}^l x - \frac{A}{\tau}\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)S_{\alpha,1}^l x$$

for all $j \geq 1$.

If $R_\alpha(t) := (g_\alpha * S_{\alpha,1})(t)$, then the Corollary 2.9 implies that $R_\alpha^j = \sum_{l=0}^{j} k_\tau^\alpha(j-l)S_{\alpha,1}^l$. Since $(g_{1-\alpha} * g_\alpha)(t) = g_1(t)$, the Corollary 2.9 implies again that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^\alpha(j-l)S_{\alpha,1}^l x &= \sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)R_\alpha^j \\
&= \int_0^\infty \rho_n^\tau(t)(g_{1-\alpha} * R_\alpha)(t)x dt \\
&= \int_0^\infty \rho_n^\tau(t)(g_1 * S_{\alpha,1})(t)x dt \\
&= \sum_{j=0}^{n} k_\tau^1(n-j)S_\alpha^j x.
\end{aligned}$$

Since $k_\tau^1(n) = \tau$ for all $n \in \mathbb{N}$, we get

$$\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^\alpha(j-l)S_{\alpha,1}^l x = \tau\sum_{j=0}^{n} S_\alpha^j x. \tag{3.36}$$

Since $\sum_{j=0}^{-l} v^j = 0$ for all $l \in \mathbb{N}$, we have

$$\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)S_{\alpha,1}^l x = \sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)S_{\alpha,1}^l x = \sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j)R_\alpha^{j-1}x,$$

and thus

$$\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)S_{\alpha,1}^l x = \tau\sum_{j=0}^{n-1} S_\alpha^j x,$$

for all $n \in \mathbb{N}$, which implies that

$$\begin{aligned}
{}_C\nabla^\alpha(S_{\alpha,1}x)^n &= \frac{A}{\tau}\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j} k_\tau^\alpha(j-l)S_{\alpha,1}^l x - \frac{A}{\tau}\sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j-1} k_\tau^\alpha(j-1-l)S_{\alpha,1}^l x \\
&= A\sum_{j=0}^{n} S_\alpha^j x - A\sum_{j=0}^{n-1} S_\alpha^j x \\
&= AS_\alpha^n x,
\end{aligned}$$

for all $n \in \mathbb{N}$ and $x \in X$, which allows us to conclude that

$${}_C\nabla^\alpha S_{\alpha,1}^n u_0 = AS_{\alpha,1}^n u_0.$$

On the other hand,

$$\begin{aligned}
{}_C\nabla^\alpha((S_{\alpha,\alpha} \star f)^n) &= \nabla_\tau^{-(1-\alpha)}\nabla_\tau^1(S_{\alpha,\alpha} \star f)^n \\
&= \sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\nabla_\tau^1(S_{\alpha,\alpha} \star f)^j \\
&= \frac{1}{\tau}\sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)(S_{\alpha,\alpha} \star f)^j - \frac{1}{\tau}\sum_{j=1}^{n} k_\tau^{1-\alpha}(n-j)(S_{\alpha,\alpha} \star f)^{j-1}.
\end{aligned}$$

We observe that by Proposition 2.10

$$(S_{\alpha,\alpha}\star f)^j=\sum_{l=0}^{j}S_{\alpha,\alpha}^{j-l}f^l=\frac{1}{\tau}\int_0^\infty \rho_j^\tau(t)(S_{\alpha,\alpha}*f)(t)dt=\frac{1}{\tau}(S_{\alpha,\alpha}*f)^j. \tag{3.37}$$

Since

$$S_{\alpha,\alpha}(t)x=g_\alpha(t)x+A(g_\alpha*S_{\alpha,\alpha})(t)x,$$

for all $x\in X$, and $t\geq 0$, we obtain

$$(S_{\alpha,\alpha}*f)(t)=(g_\alpha*f)(t)+A(g_\alpha*S_{\alpha,\alpha}*f)(t),$$

and, multiplying this equation by $\rho_j^\tau(t)$ and integrating over $[0,\infty)$ we get by Proposition 2.10 that

$$\tau(S_{\alpha,\alpha}\star f)^j=\int_0^\infty \rho_j^\tau(t)(S_{\alpha,\alpha}*f)(t)dt=\int_0^\infty \rho_j^\tau(t)(g_\alpha*f)(t)dt+A\int_0^\infty \rho_j^\tau(t)(g_\alpha*S_{\alpha,\alpha}*f)(t)dt,$$

and by Corollary 2.9 we have

$$(S_{\alpha,\alpha}\star f)^j=\frac{1}{\tau}\left[\sum_{l=0}^{j}k_\tau^\alpha(j-l)f^l+A\sum_{l=0}^{j}k_\tau^\alpha(j-l)(S_{\alpha,\alpha}*f)^l\right]. \tag{3.38}$$

The equation (3.38) implies that

$$\begin{aligned}
{}_C\nabla^\alpha((S_{\alpha,\alpha}\star f)^n) &= \frac{1}{\tau^2}\sum_{j=0}^{n}k_\tau^{1-\alpha}(n-j)\left[\sum_{l=0}^{j}k_\tau^\alpha(j-l)f^l+A\sum_{l=0}^{j}k_\tau^\alpha(j-l)(S_{\alpha,\alpha}*f)^l\right]\\
&\quad -\frac{1}{\tau^2}\sum_{j=1}^{n}k_\tau^{1-\alpha}(n-j)\left[\sum_{l=0}^{j-1}k_\tau^\alpha(j-1-l)f^l+A\sum_{l=0}^{j-1}k_\tau^\alpha(j-1-l)(S_{\alpha,\alpha}*f)^l\right].
\end{aligned}$$

Analogously to equation (3.36), we can prove that

$$\sum_{j=0}^{n}k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j}k_\tau^\alpha(j-l)f^l=\tau\sum_{j=0}^{n}f^j,\quad \sum_{j=1}^{n}k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j-1}k_\tau^\alpha(j-1-l)f^l=\tau\sum_{j=0}^{n-1}f^j,$$

$$\sum_{j=0}^{n}k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j}k_\tau^\alpha(j-l)(S_{\alpha,\alpha}*f)^l x=\tau\sum_{j=0}^{n}(S_{\alpha,\alpha}*f)^l$$

and

$$\sum_{j=1}^{n}k_\tau^{1-\alpha}(n-j)\sum_{l=0}^{j-1}k_\tau^\alpha(j-1-l)(S_{\alpha,\alpha}*f)^l x=\tau\sum_{j=0}^{n-1}(S_{\alpha,\alpha}*f)^l,$$

which implies, by (3.37), that

$${}_C\nabla^\alpha((S_{\alpha,\alpha}\star f)^n)=\frac{1}{\tau}f^n+\frac{1}{\tau}A(S_{\alpha,\alpha}*f)^n=\frac{1}{\tau}f^n+A(S_{\alpha,\alpha}\star f)^n,$$

for all $n\in\mathbb{N}$. We conclude that if $u^n:=S_{\alpha,1}^n u_0+\tau(S_{\alpha,\alpha}\star f)^n$, then

$${}_C\nabla^\alpha(u^n)={}_C\nabla^\alpha\left(S_{\alpha,1}^n u_0+\tau(S_{\alpha,\alpha}\star f)^n\right)=AS_{\alpha,1}^n u_0+\tau A(S_{\alpha,\alpha}\star f)^n+f^n=Au^n+f^n,$$

for all $n\in\mathbb{N}$, that is, u^n solves the equation

$${}_C\nabla^\alpha u^n=Au^n+f^n,\quad n\in\mathbb{N}.$$

The uniqueness, follows from the uniqueness of the resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ generated by A. $\square$

If A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then two subordination principles show that A generates an $(\alpha,1)$-resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ (see [7, Theorem 3.1]) and an (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ (see [30, Theorem 3.1]), given, respectively by

$$S_{\alpha,1}(t)x=\int_0^\infty \Phi_\alpha(r)T(rt^\alpha)xdr \quad \text{and} \quad S_{\alpha,\alpha}(t)x=\alpha\int_0^\infty t^{\alpha-1}r\Phi_\alpha(r)T(rt^\alpha)xdr \quad t\geq 0, x\in X, \tag{3.39}$$

where Φ_α is the Wright type function ([45, Appendix F])

$$\Phi_\alpha(z):=\sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\alpha n+1-\alpha)}=\int_\gamma \mu^{\alpha-1}e^{\mu-z\mu^\alpha}d\mu,$$

where γ is a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise. Therefore, we have the following result.

Corollary 3.14. *Let A be the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. If $u_0\in X$ and f is a bounded function, then the fractional difference equation*

$${}_C\nabla^\alpha u^n=Au^n+f^n, \quad n\in\mathbb{N},$$

with the initial condition $u^0=u_0$, has a unique solution given by

$$u^n=S^n_{\alpha,1}u_0+\tau(S_{\alpha,\alpha}\star f)^n,$$

for all $n\in\mathbb{N}$, where $\{S_{\alpha,1}(t)\}_{t\geq 0}$ and $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ are given in (3.39).

Remark 3.15. *We notice in Theorem 3.12 that if A is a sectorial operator which generates also an (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$, then by Theorem 3.11, $D^{n+1}_{\alpha,1}=S^n_{\alpha,1}$ and $D^{n+1}_{\alpha,\alpha}=S^n_{\alpha,\alpha}$ for all $n\in\mathbb{N}$, which means that in this case the solutions given in* (3.33) *and* (3.34) *are the same.*

3.2. **Riemann-Liouville fractional difference equation.** Now, we consider the fractional difference equation for the Riemann-Liouville difference operator

$$({}^R\nabla^\alpha u)^n=Au^n+f^n, \quad n\in\mathbb{N}.$$

Thanks to the properties of $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ given in Proposition 2.2 and Theorem 2.7, the proof of the next result follows similarly to the proof of Theorem 3.13. We omit the details.

Theorem 3.16. *Let A be the generator of an (α,α)-resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ in a Banach space X. If $u^0\in X$ and f is a bounded function, then the fractional difference equation*

$$({}^R\nabla^\alpha u)^n=Au^n+f^n, \quad n\in\mathbb{N},$$

with the initial condition $u^0=u_0$, has a unique solution given by

$$u^n=S^n_{\alpha,\alpha}u_0+\tau(S_{\alpha,\alpha}\star f)^n, \tag{3.40}$$

for all $n\in\mathbb{N}$.

The proof of the next result follows similarly to the proof of Theorem 3.12. We omit the details.

Theorem 3.17. *Let A be a sectorial operator in a Banach space X. If $u_0\in\ker(A)$ and $f^0=0$, then the fractional difference equation*

$${}^R\nabla^\alpha u^n=Au^n+f^n, \quad n\in\mathbb{N}, \tag{3.41}$$

with the initial condition $u^0=u_0$, has a unique solution given by

$$u^n=D^{n+1}_{\alpha,\alpha}u_0+\tau\sum_{j=0}^n D^{n+1-j}_{\alpha,\alpha}f^j,$$

for all $n\in\mathbb{N}$.

4. Error estimates for sectorial operators

In this section we study the convergence of the method introduced in Section 3 and we compare the solution u^n to the equation (3.32) and the solution u to the equation (1.3) at time $t_n = n\tau$, where $\tau > 0$. We prove here the result for the Caputo fractional derivative. The Riemann-Liouville case is analogous and we omit the details.

For a closed operator $A \in \mathrm{Sec}(\theta, M)$, we will consider the following path Γ_t : For $\frac{\pi}{2} < \theta < \pi$, we take ϕ such that $\frac{1}{2}\phi < \frac{\pi}{2}\alpha < \phi < \theta$. Next, we define Γ_t (see Figure 1) as the union $\Gamma_t^1 \cup \Gamma_t^2$, where

$$\Gamma_t^1 := \left\{ \frac{1}{t} e^{i\psi/\alpha} : -\phi < \psi < \phi \right\} \quad \text{and} \quad \Gamma_t^2 := \left\{ r e^{\pm i\phi/\alpha} : \frac{1}{t} \leq r \right\}.$$

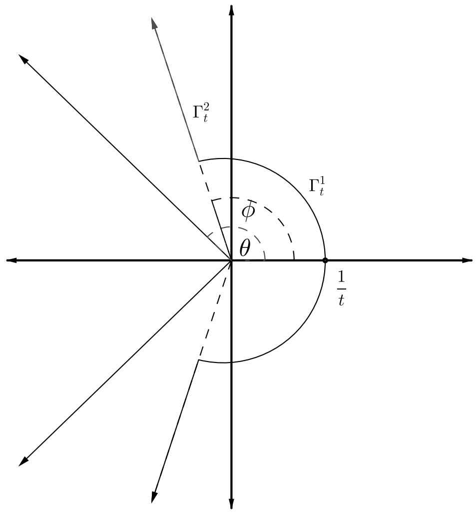

Figure 1. Plot of path Γ_t.

The next result will be useful to prove the main theorem in this section. A similar result can be found in [14].

Lemma 4.18. *Let $A \in \mathrm{Sec}(\theta, M)$ and Γ be the complex path defined above. If $\mu \geq 0$, then there exist positive constants C_α, depending only on α, such that*

$$\int_{\Gamma_t} \left| \frac{e^{zt}}{z^\mu} \right| |dz| \leq C_\alpha t^{\mu-1}$$

for all $t > 0$, where

$$C_\alpha := \left(2\phi \int_{-\phi}^{\phi} e^{\cos(\psi/\alpha)} d\psi + \frac{2}{-\cos(\phi/\alpha)} \right).$$

Proof. On Γ_t^1 we have

$$\int_{\Gamma_t^1} \left| \frac{e^{zt}}{z^\mu} \right| |dz| \leq 2\phi \int_{-\phi}^{\phi} \frac{e^{(t\frac{\cos(\psi/\alpha)}{t})}}{\left| \frac{e^{(i\mu\psi/\alpha)}}{t^\mu} \right|} \frac{1}{t} d\psi = 2\phi \int_{-\phi}^{\phi} e^{\cos(\psi/\alpha)} d\psi t^{\mu-1}.$$

On the other hand, since $\frac{1}{2}\phi < \frac{\pi}{2}\alpha < \phi$ we obtain $\frac{\pi}{2} < \frac{\phi}{\alpha} < \pi$, and thus $\cos(\phi/\alpha) < 0$, which implies that on Γ_t^2 we have

$$\int_{\Gamma_t^2} \left| \frac{e^{zt}}{z^\mu} \right| |dz| \leq 2 \int_{\frac{1}{t}}^{\infty} \frac{e^{rt\cos(\phi/\alpha)}}{r^\mu} dr \leq 2t^\mu \int_0^\infty e^{rt\cos(\phi/\alpha)} dr = 2\frac{t^{\mu-1}}{-\cos(\phi/\alpha)}.$$

We conclude that

$$\int_{\Gamma_t}\left|\frac{e^{zt}}{z^\mu}\right||dz|\le\left(2\phi\int_{-\phi}^{\phi}e^{\cos(\psi/\alpha)}d\psi+\frac{2}{-\cos(\phi/\alpha)}\right)t^{\mu-1}.$$

□

Now, we notice that if $z=\frac{1}{t}e^{i\phi/\alpha}$, then $z^\alpha=\frac{1}{t^\alpha}e^{i\phi}$ and $\arg(z^\alpha)=\phi<\theta$, which means that $z^\alpha\in\rho(A)$, because $A\in\mathrm{Sec}(\theta,M)$. Hence, by using the path $\Gamma\equiv\Gamma_t$, the inversion formula of the Laplace transform allows to write

$$S_{\alpha,\beta}(t)=\frac{1}{2\pi i}\int_\Gamma e^{zt}z^{\alpha-\beta}(z^\alpha-A)^{-1}dz,\quad t>0. \tag{4.42}$$

Given $0<\varepsilon<1$, the space of all continuous function $f:[0,\infty)\to D(A^\varepsilon)$ endowed with the norm $\|f\|_\varepsilon:=\sup_{t\ge0}\|f(t)\|_\varepsilon=\sup_{t\ge0}\|A^\varepsilon f(t)\|$ will be denoted by $C([0,\infty),D(A^\varepsilon))$.

Theorem 4.19. *Let $0<\alpha,\varepsilon<1$ such that $1<\alpha(1+\varepsilon)<2$. Suppose that $f\in C([0,\infty),D(A^\varepsilon))$. Let $A\in\mathrm{Sec}(\theta,M)$ which generates an (α,α) resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\ge0}$. Let Γ be the complex path defined above. If $u_0\in D(A^\varepsilon)$, then for each $T>0$ there exists a constant $C=C(T)>0$ (independent of the solution, the data and the step size) such that, for $0<t_n\le T$, there holds*

$$\|u^n-u(t_n)\|\le C\tau t_n^{\alpha\varepsilon-1}\left(\|u_0\|_\varepsilon+\|f\|_\varepsilon\right). \tag{4.43}$$

Proof. The solution to (1.1) is given by

$$u(t)=S_{\alpha,1}(t)u_0+(S_{\alpha,\alpha}*f)(t),$$

and by Theorem 3.13, the solution to (3.32) is given by

$$u^n=S^n_{\alpha,1}u_0+\tau(S_{\alpha,\alpha}\star f)^n.$$

Fix $n\in\mathbb{N}$ such that $0<t_n\le T$, where $t_n:=\tau n$. Then, we have

$$\|u^n-u(t_n)\|\le\|(S_{\alpha,1}(t_n)-S^n_{\alpha,1})u_0\|+\|(S_{\alpha,\alpha}*f)(t_n)-\tau(S_{\alpha,\alpha}\star f)^n\|:=I_1+I_2. \tag{4.44}$$

To estimate the first term, we notice that

$$I_1\le\int_0^\infty\rho_n^\tau(t)\|(S_{\alpha,1}(t)-S_{\alpha,1}(t_n))u_0\|dt.$$

Now, if $\Gamma=\Gamma_{t_n}$ then, by (4.42) we can write

$$\begin{aligned}(S_{\alpha,1}(t)-S_{\alpha,1}(t_n))u_0&=\frac{1}{2\pi i}\int_\Gamma(e^{zt}-e^{zt_n})z^{\alpha-1}(z^\alpha-A)^{-1}u_0dz\\&=\frac{1}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zt_n})}{z}z^\alpha(z^\alpha-A)^{-1}u_0dz.\end{aligned}$$

Since

$$z^\alpha(z^\alpha-A)^{-1}=(z^\alpha-A)^{-1}A+I=A(z^\alpha-A)^{-1}+I\ \text{ and }\ A(z^\alpha-A)^{-1}=A^{1-\varepsilon}(z^\alpha-A)^{-1}A^\varepsilon, \tag{4.45}$$

we have

$$(S_{\alpha,1}(t)-S_{\alpha,1}(t_n))u_0=\frac{1}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zt_n})}{z}u_0dz+\frac{1}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zt_n})}{z}A^{1-\varepsilon}(z^\alpha-A)^{-1}A^\varepsilon u_0dz.$$

Since $h(z):=\frac{(e^{zt}-e^{zt_n})}{z}$ has a unique removable singularity at $z=0$ and $t\ge t_n$, $h(z)$ can be analytically extended to the region enclosed by the path $\Gamma^R:=\Gamma^R_{t_n}$ where Γ^R is the path given in Figure 2, and therefore

$$\frac{1}{2\pi i}\int_{\Gamma^R}\frac{(e^{zt}-e^{zt_n})}{z}u_0dz=0.$$

Since

$$\frac{1}{2\pi i}\int_\Gamma\frac{(e^{zt}-e^{zt_n})}{z}u_0dz=\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma^R}\frac{(e^{zt}-e^{zt_n})}{z}u_0dz,$$

we obtain

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{(e^{zt}-e^{zt_n})}{z}u_0 dz = 0.$$

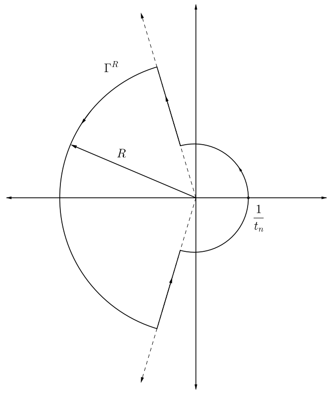

FIGURE 2. Plot of path Γ^R.

On the other hand, since A is a sectorial operator, we get by (2.10)

$$\|A^{1-\varepsilon}(z^\alpha - A)^{-1}x\| \le K(M+1)^{1-\varepsilon}\frac{\|x\|}{|z|^{\alpha\varepsilon}},$$

for all $x \in X$, which implies that

$$\|(S_{\alpha,1}(t)-S_{\alpha,1}(t_n))u_0\| \le \frac{K(M+1)^{1-\varepsilon}}{2\pi}\int_{\Gamma}\frac{|e^{zt}-e^{zt_n}|}{|z|}\frac{\|A^\varepsilon u_0\|}{|z|^{\alpha\varepsilon}}|dz|.$$

By the mean value for complex valued functions, there exist t_0, t_1 with $0 < t_n < t_0 < t_1 < t$ such that

$$\frac{|e^{zt}-e^{zt_n}|}{|z|} \le (t-t_n)\left(|e^{t_0 z}|+|e^{t_1 z}|\right).$$

Therefore, by Lemma 4.18 we obtain

$$\begin{aligned}\|(S_{\alpha,1}(t)-S_{\alpha,1}(t_n))u_0\| &\le \frac{K(M+1)^{1-\varepsilon}(t-t_n)\|A^\varepsilon u_0\|}{2\pi}\int_{\Gamma}\frac{|e^{zt_0}|+|e^{zt_1}|}{|z|^{\alpha\varepsilon}}|dz| \\ &\le \frac{K(M+1)^{1-\varepsilon}(t-t_n)}{2\pi}C_\alpha(t_0^{\alpha\varepsilon-1}+t_1^{\alpha\varepsilon-1})\|A^\varepsilon u_0\|.\end{aligned}$$

Since $0 < \alpha, \varepsilon < 1$ and $t_n < t_0 < t_1$ we obtain $t_1^{\alpha\varepsilon-1} < t_0^{\alpha\varepsilon-1} < t_n^{\alpha\varepsilon-1}$ and thus

$$\|(S_{\alpha,1}(t)-S_{\alpha,1}(t_n))u_0\| \le \frac{K(M+1)^{1-\varepsilon}(t-t_n)}{\pi}C_\alpha t_n^{\alpha\varepsilon-1}\|u_0\|_\varepsilon.$$

We conclude that

$$\int_0^\infty \rho_n^\tau(t)\|(S_{\alpha,1}(t)-S_{\alpha,1}(t_n))u_0\|dt \le \frac{K(M+1)^{1-\varepsilon}}{\pi}C_\alpha t_n^{\alpha\varepsilon-1}\|A^\varepsilon u_0\|\int_0^\infty \rho_n^\tau(t)(t-t_n)dt.$$

Since $\int_0^\infty \rho_n^\tau(t)dt = 1$ for all $n \in \mathbb{N}$, an easy computation shows that

$$\int_0^\infty \rho_n^\tau(t)(t-t_n)dt = \int_0^\infty \rho_n^\tau(t)tdt - t_n = t_{n+1}-t_n = \tau, \tag{4.46}$$

which implies that

$$\int_0^\infty \rho_n^\tau(t)\|(S_{\alpha,1}(t)-S_{\alpha,1}(t_n))u_0\|dt \le \frac{KC_\alpha(M+1)^{1-\varepsilon}}{\pi}t_n^{\alpha\varepsilon-1}\|u_0\|_\varepsilon\tau.$$

On the other hand, the second term in (4.44) can be written, by Proposition 2.10, as

$$\|(S_{\alpha,\alpha}*f)(t_n)-\tau(S_{\alpha,\alpha}\star f)^n\| \quad = \quad \left\|\int_0^\infty \rho_n^\tau(t)[(S_{\alpha,\alpha}*f)(t_n)-(S_{\alpha,\alpha}*f)(t)]dt\right\|.$$

Moreover, we can write

$$\begin{aligned}(S_{\alpha,\alpha}*f)(t)-(S_{\alpha,\alpha}*f)(t_n) &= \int_0^t S_{\alpha,\alpha}(t-r)f(r)dr-\int_0^{t_n} S_{\alpha,\alpha}(t_n-r)f(r)dr\\ &= \int_0^{t_n}(S_{\alpha,\alpha}(t-r)-S_{\alpha,\alpha}(t_n-r))f(r)dr+\int_{t_n}^t S_{\alpha,\alpha}(t-r)f(r)dr\\ &:= J_1+J_2.\end{aligned}$$

Now, we estimate the integrals J_1 and J_2. In order to estimate J_1 we estimate, for simplicity, the norm $\|(S_{\alpha,\alpha}(t)-S_{\alpha,\alpha}(s))x\|$ for $x\in D(A^\varepsilon)$, with $t>s>0$. By using the path Γ, we can write

$$(S_{\alpha,\alpha}(t)-S_{\alpha,\alpha}(s))x=\frac{1}{2\pi i}\int_\Gamma (e^{zt}-e^{zs})(z^\alpha-A)^{-1}xdz.$$

By (4.45) we have

$$\begin{aligned}(S_{\alpha,\alpha}(t)-S_{\alpha,\alpha}(s))x &= \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt}-e^{zs})}{z^\alpha}z^\alpha(z^\alpha-A)^{-1}xdz\\ &= \frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt}-e^{zs})}{z^\alpha}A^{1-\varepsilon}(z^\alpha-A)^{-1}A^\varepsilon xdz+\frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt}-e^{zs})}{z^\alpha}xdz.\end{aligned}$$

Since $0<\alpha<1$, the function

$$g(z):=\frac{(e^{zt}-e^{zs})}{z^\alpha}=\frac{(e^{zt}-e^{zs})}{z}z^{1-\alpha}$$

has a unique removable singularity at $z=0$ and thus

$$\frac{1}{2\pi i}\int_\Gamma \frac{(e^{zt}-e^{zs})}{z^\alpha}xdz=0,$$

(following the same method used to prove that $\int_\Gamma h(z)u_0dz=0$). Next, we notice that if $x\in D(A^\varepsilon)$, then the sectoriality of operator A implies that

$$\|A^{1-\varepsilon}(z^\alpha-A)^{-1}A^\varepsilon x\|\le K(M+1)^{1-\varepsilon}\frac{\|A^\varepsilon x\|}{|z|^{\alpha\varepsilon}}, \tag{4.47}$$

which gives

$$\begin{aligned}\|(S_{\alpha,\alpha}(t)-S_{\alpha,\alpha}(s))x\| &\le \frac{1}{2\pi}K(M+1)^{1-\varepsilon}\int_\Gamma \frac{|e^{zt}-e^{zs}|}{|z|^\alpha}\frac{\|A^\varepsilon x\|}{|z|^{\alpha\varepsilon}}|dz|\\ &= \frac{1}{2\pi}K(M+1)^{1-\varepsilon}\int_\Gamma \frac{|e^{zt}-e^{zs}|}{|z|}\frac{1}{|z|^{\alpha(\varepsilon+1)-1}}|dz|\|A^\varepsilon x\|.\end{aligned}$$

The mean value theorem implies the existence of t_0,t_1 with $0<s<t_0<t_1<t$ such that

$$\frac{|e^{zt}-e^{zs}|}{|z|}\le(t-s)\left(|e^{t_0z}|+|e^{t_1z}|\right).$$

Since $0 < \alpha(1+\varepsilon) - 1 < 1$ and $s < t_0 < t_1 < t$ we have $t_1^{\alpha(1+\varepsilon)-2} < t_0^{\alpha(1+\varepsilon)-2} < s^{\alpha(1+\varepsilon)-2}$. By Lemma 4.18 we get

$$\begin{aligned}
\|(S_{\alpha,\alpha}(t) - S_{\alpha,\alpha}(s))x\| &\leq \frac{1}{2\pi} K(M+1)^{1-\varepsilon}(t-s)\int_\Gamma \frac{|e^{t_0 z}| + |e^{t_1 z}|}{|z|^{\alpha(\varepsilon+1)-1}}|dz|\|A^\varepsilon x\| \\
&\leq \frac{C}{2\pi} K(M+1)^{1-\varepsilon}(t-s)(t_0^{\alpha(1+\varepsilon)-2} + t_1^{\alpha(1+\varepsilon)-2})\|A^\varepsilon x\| \\
&\leq \frac{C}{\pi} K(M+1)^{1-\varepsilon}(t-s)s^{\alpha(1+\varepsilon)-2}\|A^\varepsilon x\|.
\end{aligned}$$

Replacing t by $t-r$ and s by $t_n - r$ we can estimate J_1 as

$$\begin{aligned}
\int_0^{t_n} \|(S_{\alpha,\alpha}(t-r) - S_{\alpha,\alpha}(t_n - r))f(r)\|dr &\leq \frac{C}{\pi} K(M+1)^{1-\varepsilon}(t-t_n)\int_0^{t_n}(t_n - r)^{\alpha(1+\varepsilon)-2}\|A^\varepsilon f(r)\|dr \\
&\leq \frac{C\|f\|_\varepsilon}{\pi} K(M+1)^{1-\varepsilon}(t-t_n)\int_0^{t_n}(t_n - r)^{\alpha(1+\varepsilon)-2}dr \\
&= \frac{C_\alpha\|f\|_\varepsilon}{\pi} K(M+1)^{1-\varepsilon}(t-t_n)g_{\alpha(1+\varepsilon)}(t_n)\Gamma(\alpha(1+\varepsilon)-1) \\
&= \frac{C_\alpha\|f\|_\varepsilon}{\pi} K(M+1)^{1-\varepsilon}(t-t_n)t_n^{\alpha(1+\varepsilon)-1}\frac{\Gamma(\alpha(1+\varepsilon)-1)}{\Gamma(\alpha(1+\varepsilon))} \\
&= \frac{C_\alpha\|f\|_\varepsilon}{\pi} K(M+1)^{1-\varepsilon}(t-t_n)t_n^{\alpha(1+\varepsilon)-1}\frac{1}{\alpha(1+\varepsilon)-1}.
\end{aligned}$$

Since

$$\int_0^\infty \rho_n^\tau(t)(t-t_n)t_n^{\alpha(1+\varepsilon)-1}dt = t_{n+1}t_n^{\alpha(1+\varepsilon)-1} - t_n^{\alpha(1+\varepsilon)} = \tau t_n^{\alpha(1+\varepsilon)-1} \tag{4.48}$$

we have

$$\begin{aligned}
\int_0^\infty \rho_n^\tau(t)\left\|\int_0^{t_n}(S_{\alpha,\alpha}(t-r) - S_{\alpha,\alpha}(t_n - r))f(r)dr\right\|dt &\leq \frac{C_\alpha\|f\|_\varepsilon}{\pi} K(M+1)^{1-\varepsilon}\frac{1}{\alpha(1+\varepsilon)-1}\tau t_n^{\alpha(1+\varepsilon)-1} \\
&\leq \frac{C_\alpha\|f\|_\varepsilon}{\pi} T^\alpha K(M+1)^{1-\varepsilon}\frac{1}{\alpha(1+\varepsilon)-1}\tau t_n^{\alpha\varepsilon-1}.
\end{aligned}$$

Finally, we estimate the integral J_2. First, by (4.45) we can write

$$S_{\alpha,\alpha}(t)x = \frac{1}{2\pi i}\int_\Gamma e^{zt}(z^\alpha - A)^{-1}xdz = \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt}}{z^\alpha}xdz + \frac{1}{2\pi i}\int_\Gamma \frac{e^{zt}}{z^\alpha}A^{1-\varepsilon}(z^\alpha - A)^{-1}A^\varepsilon xdz$$

for all $x \in D(A^\varepsilon)$. The first integral in the last equality is equal to zero, because the function $z \mapsto \frac{e^{tz}}{z^\alpha} = \frac{e^{tz}}{z}z^{1-\alpha}$ has a unique removable singulary at $z = 0$. On the other hand, by (4.47) and Lemma 4.18 we obtain

$$\|S_{\alpha,\alpha}(t)x\| \leq \frac{K(M+1)^{1-\varepsilon}}{2\pi}\int_\Gamma \frac{|e^{zt}|}{|z|^{\alpha(1+\varepsilon)}}\|A^\varepsilon x\||dz| \leq \frac{KC_\alpha(M+1)^{1-\varepsilon}}{2\pi}t^{\alpha(1+\varepsilon)-1}\|A^\varepsilon x\|,$$

for all $t > 0$ and $x \in D(A^\varepsilon)$. Therefore,

$$\int_{t_n}^t \|S_{\alpha,\alpha}(t-r)f(r)\|dr \leq \frac{KC_\alpha\|f\|_\varepsilon(M+1)^{1-\varepsilon}}{2\pi}\int_{t_n}^t (t-r)^{\alpha(1+\varepsilon)-1}dr.$$

We observe that

$$\int_{t_n}^t (t-r)^{\alpha(1+\varepsilon)-1}dr = \int_0^t (t-r)^{\alpha(1+\varepsilon)-1}dr - \int_0^{t_n}(t-r)^{\alpha(1+\varepsilon)-1}dr.$$

Since

$$\int_0^t (t-r)^{\alpha(1+\varepsilon)-1}dr = \Gamma(\alpha(1+\varepsilon))(g_1 * g_{\alpha(1+\varepsilon)})(t) = \Gamma(\alpha(1+\varepsilon))g_{\alpha(1+\varepsilon)+1}(t) = \frac{1}{\alpha(1+\varepsilon)}t^{\alpha(1+\varepsilon)},$$

for all $t \geq 0$, $x \mapsto x^{\alpha(1+\varepsilon)-1}$ is an increasing function and $t_n \leq t$, we obtain

$$\int_{t_n}^{t} (t-r)^{\alpha(1+\varepsilon)-1} dr = \frac{1}{\alpha(1+\varepsilon)} t^{\alpha(1+\varepsilon)} - \int_0^{t_n} (t-r)^{\alpha(1+\varepsilon)-1} dr \leq \frac{1}{\alpha(1+\varepsilon)} (t^{\alpha(1+\varepsilon)} - t_n^{\alpha(1+\varepsilon)}).$$

We conclude that

$$\int_{t_n}^{t} \|S_{\alpha,\alpha}(t-r)f(r)\| dr \leq \frac{KC_\alpha \|f\|_\varepsilon (M+1)^{1-\varepsilon}}{2\pi} \frac{1}{\alpha(1+\varepsilon)} (t^{\alpha(1+\varepsilon)} - t_n^{\alpha(1+\varepsilon)}),$$

for all $t \geq 0$. An easy computation shows that

$$\int_0^\infty \rho_n^\tau(t) t^{\alpha(1+\varepsilon)} dt = \frac{\tau^{\alpha(1+\varepsilon)}}{n!} \Gamma(n+1+\alpha(1+\varepsilon)).$$

Since for $0 < \lambda < 1$ and $n \geq 0$, $\frac{\Gamma(n+\lambda)}{\Gamma(n+1)} < n^{\lambda-1}$ (see for instance [19]), and $0 < \alpha(1+\varepsilon)-1 < 1$ we obtain

$$\begin{aligned} \frac{\tau^{\alpha(1+\varepsilon)}}{n!} \Gamma(n+1+\alpha(1+\varepsilon)) &= \tau\tau^{\alpha(1+\varepsilon)-1} \frac{\Gamma(n+1+\alpha(1+\varepsilon)-1)}{\Gamma(n+2)} (n+1)(n+\alpha(1+\varepsilon)) \\ &< t_n t_{n+1} t_{n+1}^{\alpha(1+\varepsilon)-2} + \alpha(1+\varepsilon)\tau t_{n+1}^{\alpha(1+\varepsilon)-1} \end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, the function $x \mapsto x^{\alpha(1+\varepsilon)-2}$ is a decreasing function, which implies that $t_{n+1}^{\alpha(1+\varepsilon)-2} \leq t_n^{\alpha(1+\varepsilon)-2}$ for all $n \in \mathbb{N}$, and thus

$$\frac{\tau^{\alpha(1+\varepsilon)}}{n!} \Gamma(n+1+\alpha(1+\varepsilon)) \leq t_{n+1} t_n^{\alpha(1+\varepsilon)-1} + \alpha(1+\varepsilon)\tau t_{n+1}^{\alpha(1+\varepsilon)-1}.$$

By (4.48), we get

$$\int_0^\infty \rho_n^\tau(t)(t^{\alpha(1+\varepsilon)} - t_n^{\alpha(1+\varepsilon)}) dt = \frac{\tau^{\alpha(1+\varepsilon)}}{n!} \Gamma(n+1+\alpha(1+\varepsilon)) - t_n^{\alpha(1+\varepsilon)} \leq \tau t_n^{\alpha(1+\varepsilon)-1} + \alpha(1+\varepsilon)\tau t_{n+1}^{\alpha(1+\varepsilon)-1}.$$

Since $t_{n+1}^{\alpha(1+\varepsilon)-1} = (n+1)\tau t_{n+1}^{\alpha(1+\varepsilon)-2} \leq t_n^{\alpha(1+\varepsilon)-1} + \tau t_{n+1}^{\alpha(1+\varepsilon)-2} \leq 2t_n^{\alpha(1+\varepsilon)-1}$ for all $n \in \mathbb{N}$, and $0 < t_n \leq T$ we obtain

$$\int_0^\infty \rho_n^\tau(t)(t^{\alpha(1+\varepsilon)} - t_n^{\alpha(1+\varepsilon)}) dt \leq (1+2\alpha(1+\varepsilon))T^\alpha \tau t_n^{\alpha\varepsilon-1},$$

and we conclude that

$$\begin{aligned} \int_0^\infty \rho_n^\tau(t) \int_{t_n}^t \|S_{\alpha,\alpha}(t-r)f(r)\| dr &\leq \frac{KC_\alpha \|f\|_\varepsilon (M+1)^{1-\varepsilon}}{2\pi} \frac{1}{\alpha(1+\varepsilon)} \int_0^\infty \rho_n^\tau(t)(t^{\alpha(1+\varepsilon)} - t_n^{\alpha(1+\varepsilon)}) dt \\ &\leq \frac{KC_\alpha \|f\|_\varepsilon (M+1)^{1-\varepsilon}}{\pi} \frac{(1+2\alpha(1+\varepsilon))T^\alpha}{\alpha(1+\varepsilon)} \tau t_n^{\alpha\varepsilon-1}. \end{aligned}$$

Finally, the constant $C = C(T)$ is given by

$$C = \frac{KC_\alpha (M+1)^{1-\varepsilon}}{\pi} \max\left\{1, \frac{T^\alpha}{\alpha(1+\varepsilon)-1}, \frac{(1+2\alpha(1+\varepsilon))T^\alpha}{\alpha(1+\varepsilon)}\right\}.$$

□

5. Numerical Experiments

In this section we give some numerical experiments on the behavior of the Caputo fractional derivative ${}_C\partial_t^\alpha$ and the Caputo difference operator $({}_C\nabla^\alpha)^n$. Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. We assume that $t \in [0,1]$ and we take $\tau = \frac{1}{N}$, where $N \in \mathbb{N}$ is given. We remark that, $n!$ cannot be represented as a 64-bit integer for $n > 20$ or as a 64-bit floating-point number for $n > 171$. The Stirling approximation formula $n! \simeq \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$ can be used to calculate $n!$ for large values of n (see for instance [55, Chapter 2]). On the other hand, since $\int_0^\infty \rho_n^\tau(t) dt = 1$, we notice that if f has a bounded derivative, then by (4.46) we have $\|f^n - f(t_n)\| = \left\|\int_0^\infty \rho_n^\tau(t)(f(t) - f(t_n)) dt\right\| = \left\|\int_0^\infty \rho_n^\tau(t) \left(\int_{t_n}^t f'(s) ds\right) dt\right\| \leq \tau \|f'\|_\infty$, and then, in the numerical experiments, $f^n = \int_0^\infty \rho_n^\tau(t) f(t) dt$ can be computed by using $f(t_n) = f(n\tau)$.

We first consider the function $u(t) := t^p$, where $p > m$. By [31, Chapter 2, Property 2.1] we have ${}_C\partial_t^\alpha u(t) = \frac{\Gamma(p+1)t^{p-\alpha}}{\Gamma(p-\alpha+1)}$. If $\alpha = \frac{1}{2}$, and $p = 4$, then $m = 1$ and ${}_C\partial_t^{1/2} u(t) = \frac{128t^{7/2}}{35\sqrt{\pi}}$. Moreover,

$$({}_C\nabla^\alpha u)^n = \nabla_\tau^{-(1-\alpha)}(\nabla_\tau^1 u)^n = \sum_{j=0}^{n} k_\tau^{1-\alpha}(n-j)\frac{u^j - u^{j-1}}{\tau} = \frac{\tau^{\frac{1}{2}}}{\sqrt{\pi}}\sum_{j=0}^{n}\frac{\Gamma(\frac{1}{2}+n-j)}{\Gamma(n-j+1)}\frac{u^j-u^{j-1}}{\tau}$$

for all $n \in \mathbb{N}_0$, where $u^j = t_j^p$. In Figure 3, we have the Caputo fractional derivative ${}_C\partial_t^\alpha u(t)$ and the Caputo difference operator $({}_C\nabla^\alpha u)^n$ on the interval $[0, 1]$, where $0 \le n \le N$ and $N = 20, 50$ and $N = 100$, respectively.

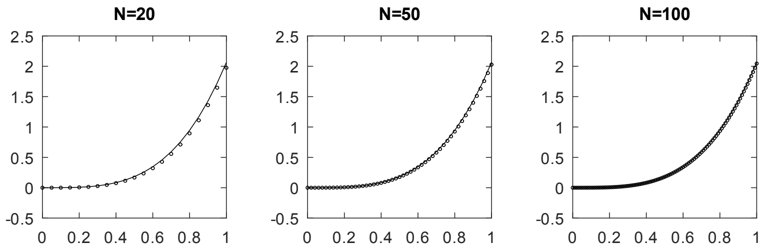

FIGURE 3. ${}_C\partial_t^\alpha u(t)$ (line) and $({}_C\nabla^\alpha u)^n$ (circles) for $N = 20, N = 50$ and $N = 100$.

Now, we illustrate the exact $u(t)$ and the approximated solution u^n (given in Theorem 3.13) to a fractional differential and difference equations of order $\alpha = \frac{1}{2}$. If $f(t) = t^2e^{-t}$, then the solution to the Caputo fractional differential equation

$$\partial_t^{\frac{1}{2}} u(t) = -u(t) + t^2e^{-t}, \quad t \ge 0, \quad u(0) = 1, \tag{5.49}$$

is given by

$$u(t) = e_{\frac{1}{2},1}(t) + \int_0^t e_{\frac{1}{2},\frac{1}{2}}(t-s)s^2e^{-s}ds,$$

where $e_{\frac{1}{2},1}(t) := E_{\frac{1}{2},1}(-t^{\frac{1}{2}})$, and $e_{\frac{1}{2},\frac{1}{2}}(t) := t^{\frac{1}{2}-1}E_{\frac{1}{2},\frac{1}{2}}(-t^{\frac{1}{2}})$.

In Figure 4 we have the exact and approximated solution to (5.49) on the interval $[0, 5]$. Here, we consider $\tau = 5/N$ for $N = 20$, $N = 50$ and $N = 100$.

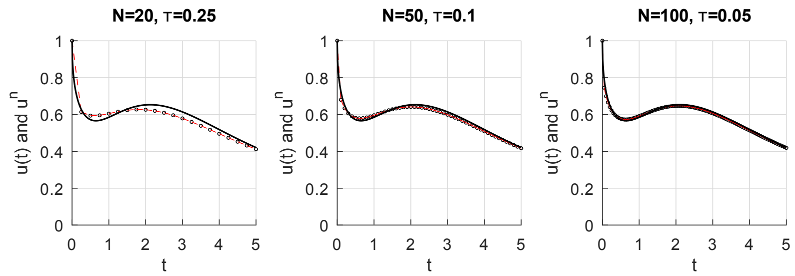

FIGURE 4. The exact solution $u(t)$ (line) and the approximated solution u^n (circles) to (5.49) for $N = 20, N = 50$ and $N = 100$.

Since the exact solution to (5.49) is given by means of Mittag-Leffler functions (defined as infinite series by (1.5)), in the numerical experiments, the exact and the approximated solution have been taken as finite sums of these power expansions (with 100 terms).

Next, in Figure 5 we have the absolute error e_n for $N = 20$, $N = 50$ and $N = 100$.

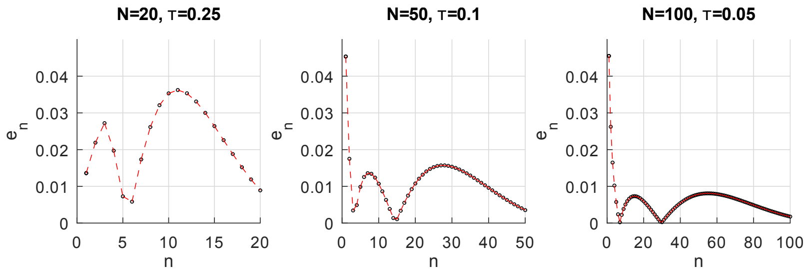

FIGURE 5. Error e_n to (5.49) for $N = 20, N = 50$ and $N = 100$. The results are in accordance to Theorem 4.19, which means that the error goes to zero as $\tau \downarrow 0$

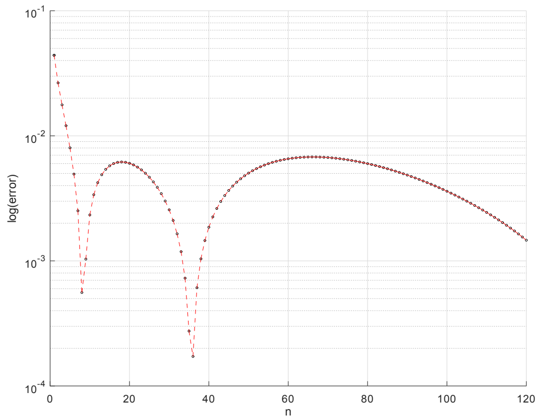

FIGURE 6. Log Error for $1 \leq n \leq 120$.

In Figure 6 we have the error as a function of t_n for $1 \leq n \leq 120$ using a log scale on the vertical axis. As before, we take $\alpha = \frac{1}{2}$.

Finally, we consider the initial value problem

$$\left\{ \begin{array}{rcll} \partial_t^\alpha u(x,t) & = & \dfrac{\partial^2 u(x,t)}{\partial x^2}, & t \geq 0, x \in [0,1], \\ u(0,t) & = & 0, & t > 0, \\ u(1,t) & = & 0, & t > 0, \\ u(x,0) & = & h_0(x), & x \in [0,1], \end{array} \right. \tag{5.50}$$

where $h_0(x) = x(1-x)$. The solution to (5.50) is given by (see for instance [54]):

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3} E_{\alpha,1}(-(2n-1)^2\pi^2 t^\alpha) \sin((2n-1)\pi x). \tag{5.51}$$

Take $\alpha = \frac{1}{2}$. In Figure 7 we have the approximated solution $u^n := u^n(x_0)$ given by

$$u^n = \int_0^\infty \rho_n^\tau(t) u(x_0,t) dt$$

for $1 \leq n \leq N$ at $x_0 = 0.3$ on the interval $[0, 0.1]$, where u is defined by (5.51). We take $N = 20, 50$ and $N = 80$. Again, in the numerical experiments, the infinite series in (5.51) are taken as finite sums of the power expansions (with 100 terms).

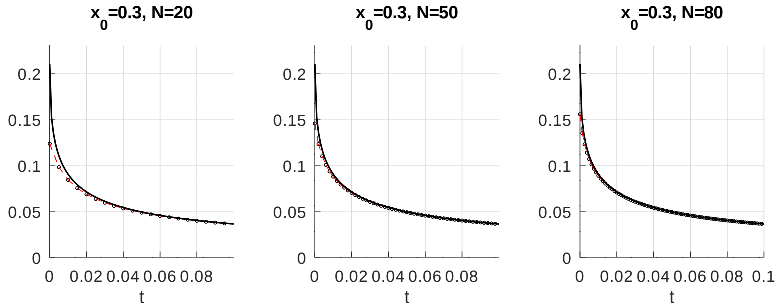

FIGURE 7. Exact (line) and approximated (circles) solution u and u^n, respectively, for $N = 20, N = 50$ and $N = 80$ at $x_0 = 0.3$.

Finally, in Figure 8 we have the approximated solution u^n for $1 \le n \le 80$ at $x_0 = 0.1$, $x_0 = 0.5$ and $x_0 = 0.9$ for t on the interval $[0, 0.1]$.

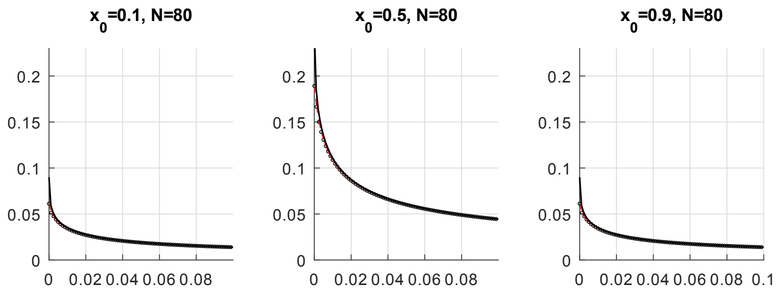

FIGURE 8. Exact (line) and approximated (circles) solution u and u^n, respectively, for $N = 80$ at $x_0 = 0.1, 0.5$ and $x_0 = 0.9$.

Acknowledgements. The author thanks to the anonymous referee for her/his carefully reading of the manuscript and for making suggestions which have improved the previous version of this paper.

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