SUBORDINATION PRINCIPLE FOR FRACTIONAL DIFFUSION-WAVE EQUATIONS OF SOBOLEV TYPE

RODRIGO PONCE

Abstract. In this paper we study subordination principles for fractional differential equations of Sobolev type in Banach space. With the help of the theory of Sobolev type resolvent families (known also as propagation family) as well as these subordination principles, we obtain the existence of mild solutions for this kind of equations. We study simultaneously the case $0 < \alpha < 1$ and $1 < \alpha < 2$ for the Caputo and Riemann-Liouville fractional derivatives.

MSC 2010: Primary 26A33; Secondary 45N05, 34A08, 47D06

Key Words and Phrases: fractional derivatives, Sobolev type differential equations, subordination principle

1. Introduction

The problem of the existence of mild solutions to fractional differential equations of Sobolev type in the form of

$$D_t^\alpha(Eu)(t) = Au(t) + Ef(t), \quad (Eu)(0) = Eu_0, \tag{1.1}$$

has been extensively studied by several authors in the last years, see for instance [13, 14, 15, 19, 22] and the references therein. Here, A and E are closed linear operators defined in a Banach space $(X, \|\cdot\|)$, u_0 belongs to $D(E)$, the domain of E, f is a suitable function satisfying $f(t) \in D(E)$ and D_t^α denotes the D_t^α for $0 < \alpha < 1$.

The change of variable $v(t) = Eu(t)$ allows to write the initial value problem (1.1) as

$$D_t^\alpha v(t) = Lv(t) + g(t), \quad v(0) = v_0, \tag{1.2}$$

where $L = AE^{-1}$, with $D(L) = E(D(A))$, $g(t) = Ef(t)$ and $v_0 = Eu_0$. Then, the mild solution to problem (1.2) is given by ([20])

$$v(t) = S_\alpha(t)v_0 + \int_0^t P_\alpha(t-s)g(s)ds,$$

where $\{S_\alpha(t)\}_{t\geq 0}$ and $\{P_\alpha(t)\}_{t\geq 0}$ are, respectively, the α-times and the α-resolvent family generated by L, whose Laplace transforms satisfy

$$\hat{S_\alpha}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha - L)^{-1} \quad \text{and} \quad \hat{P_\alpha}(\lambda) = (\lambda^\alpha - L)^{-1},$$

for λ large enough. Thus, in order to obtain the existence of a mild solution to (1.2), we just need to assume that L is the generator of an α-times resolvent family $\{S_\alpha(t)\}_{t\geq 0}$, because in this case $P_\alpha(t)=(g_{1-\alpha}*S_\alpha)(t)$ and the mild solution to (1.2) is given by

$$v(t)=S_\alpha(t)v_0+\int_0^t (g_{1-\alpha}*S_\alpha)(t-s)g(s)ds,$$

where, for $\beta>0$, $g_\beta(t):=\frac{t^{\beta-1}}{\Gamma(\beta)}$, $t>0$ and the $*$ denotes the usual finite convolution. The problem now is to find conditions on the operator L (and therefore on A and E) in order to ensure that L is the generator of an α-times resolvent family $\{S_\alpha(t)\}_{t\geq 0}$. A subordination principle ([4]) asserts that if L generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then L is the generator of the α-times resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ given by

$$S_\alpha(t)x=\int_0^\infty \Phi_\alpha(r)T(rt^\alpha)xdr, \quad t\geq 0, x\in X,$$

where Φ_α is the Wright type function ([28, Appendix F])

$$\Phi_\alpha(z):=\sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\alpha n+1-\alpha)}=\int_\gamma \mu^{\alpha-1}e^{\mu-z\mu^\alpha}d\mu,$$

where γ is a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise. Therefore, by [20, Theorem 3.1] L is also a generator of an α-resolvent family $\{P_\alpha(t)\}_{t\geq 0}$ given by

$$P_\alpha(t)x=\alpha\int_0^\infty t^{\alpha-1}r\Phi_\alpha(r)T(rt^\alpha)xdr, \quad t\geq 0, x\in X.$$

It is a well known fact (see for instance [12]) that if $D(E)\subset D(A)$, E is bijective and $E^{-1}:X\to D(E)$ is a compact operator, then $L=AE^{-1}$ is a bounded operator which generates the compact C_0-semigroup $T(t)=e^{AE^{-1}t}$, $t\geq 0$. However, in this case we need the existence (and compactness) of E^{-1} which, in general, is a restrictive assumption. In order to solve the problem of the existence of E^{-1}, more recently, the authors in [22] (see also [2]) give a subordination principle and show that if the pair (A,E) generates a Sobolev type resolvent family (also called *propagation family*, see Definition 1 below) $\{S(t)\}_{t\geq 0}$ then the pair (A,E) is also the generator of the families $\{Q(t)\}_{t\geq 0}$ and $\{R(t)\}_{t\geq 0}$ given, respectively, by

(1.3)
$$Q(t)=\int_0^\infty \xi_\alpha(r)S(t^\alpha r)dr \quad \text{and} \quad R(t)=\alpha\int_0^\infty t^{\alpha-1}r\xi_\alpha(r)S(t^\alpha r)dr$$

where $t\geq 0$ and for $r\geq 0$

$$\begin{aligned}\xi_\alpha(r) &= \frac{1}{\alpha}r^{-(1+\frac{1}{\alpha})}\varpi_\alpha(r^{-\frac{1}{\alpha}}),\\ \varpi_\alpha(r) &= \frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}r^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha).\end{aligned}$$

By [32] we notice that the function ξ_α coincides with the Wright type function Φ_α. Therefore, in this case, and without the assumption of the existence of E^{-1}, the mild solution to problem (1.1) is given by

$$u(t) = Q(t)u_0 + \int_0^t R(t-s)g(s)ds.$$

Since the pair (A, E) in the generator of a Sobolev type resolvent family $\{S(t)\}_{t\geq 0}$, there exist $\omega > 0$ and $M > 0$ such that $\|S(t)\| \leq Me^{\omega t}$ and

$$(\lambda E - A)^{-1}Ex = \int_0^\infty e^{-\lambda t}S(t)x dt,$$

for all $x \in D(E)$ and $\lambda > \omega$. Moreover, in this case, the abstract Sobolev (also called *degenerate*) Cauchy problem

$$\left\{ \begin{array}{rcll} (Eu)'(t) &=& Au(t), & t \geq 0, \\ Eu(0) &=& u_0, & \end{array} \right. \tag{1.4}$$

has a unique mild solution given by

$$u(t) = S(t)u_0,$$

see [24] for more details. Therefore, the existence of a mild solution to (1.1) is closely related to the problem of the existence of a mild solution to (1.4).

Differential equations of Sobolev type arise in several applications, such as in the motion of a uniform liquid in fissured rocks [11] or in the infiltration of water in unsaturated porous media. In such applications the operator A is typically the Laplacian operator and E is the multiplication operator by a function $m(x)$, see for instance [25, 29]. A detailed study of linear abstract Sobolev (or degenerate) type differential equations (1.4) can be found in the monographs [18] and [33].

On the other hand, fractional differential equations of Sobolev type have been widely investigated in the last years and the results are focused mainly on Caputo fractional differential equations of order $\alpha \in (0, 1]$.

Our aim in this paper is to study the existence of mild solutions to fractional diffusion equations of Sobolev type. More concretely, in this paper we consider the equations of Sobolev type

$$D_t^\alpha (Eu)(t) = Au(t) + Ef(t), \quad (Eu)(0) = Eu_0, \tag{1.5}$$

and

$$D^\alpha (Eu)(t) = Au(t) + Ef(t), \, (g_{1-\alpha} * Eu)(0) = Eu_0, \tag{1.6}$$

for $0 < \alpha < 1$; and for $1 < \alpha < 2$, the equations

$$D_t^\alpha (Eu)(t) = Au(t) + Ef(t), \quad (Eu)(0) = Eu_0, \quad (Eu)'(0) = Eu_1, \tag{1.7}$$

and

$$D^\alpha (Eu)(t) = Au(t) + Ef(t), \, (g_{2-\alpha} * Eu)(0) = Eu_0, \, (g_{2-\alpha} * Eu)'(0) = Eu_1. \tag{1.8}$$

where A and E are closed linear operators in X, D_t^α and D^α denote, respectively, the Caputo and Riemann-Liouville fractional derivatives of order α, u_0 and u_1 are the initial conditions and f is a suitable function.

In some previous works, to establish the existence of mild solutions to Sobolev type differential equations some assumptions on operators A and E are considered:

i) $D(A) \subseteq D(E)$ and A admits a continuous inverse operator A^{-1} [16, 17],
ii) $D(A) \subseteq D(E)$ and E has the bounded inverse [18],
iii) $D(E) \subseteq D(A)$ and E has the compact inverse [9, 10].

In this paper, we study the existence of mild solutions to (1.5)–(1.8) without assuming the existence of E^{-1} or it compactness as well as without any assumption on the relation between $D(A)$ and $D(E)$. Our method is based on the theory of Sobolev type resolvent families $\{S(t)\}_{t\geq 0}$ generated by the pair (A, E) (see Definition 1 below) introduced in [24] and on a new subordination principle (see Theorem 1 below) which extends some results in [1, 4, 5, 6, 7, 8, 21, 27].

We remark here that we study simultaneously fractional differential equations of Sobolev type for the Caputo and Riemann-Liouville fractional derivatives and that, to the best of our knowledge, the initial value problems (1.7) and (1.8) in the case $1 < \alpha < 2$ has not been addressed in the existing literature by using a subordination method.

The paper is organized as follows. In Section 2, we present some preliminaries on fractional calculus and Sobolev type resolvent families needed in the next sections. In Section 3, assuming that the pair (A, E) is the generator of a Sobolev type resolvent family we derive a new subordination principle. In Section 4, we study the existence of mild solutions for problems (1.5)–(1.8). Finally, in Section 5 we study some applications of the abstract results in the previous sections.

2. Fractional calculus and Sobolev type resolvent families

Let $X \equiv (X, \|\cdot\|)$ be a Banach space. The Banach space of all bounded and linear operators from X into Y is denoted by $\mathcal{B}(X, Y)$. If A is a closed linear operator on X we denote by $\rho(A)$ the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ the resolvent operator of A defined for all $\lambda \in \rho(A)$. By $[D(A)]$ we denote the domain of A equipped with the graph norm.

The strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be *exponentially bounded* if there exist $M > 0$ and $w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{wt}$, for all $t > 0$.

Definition 1. *Let A, E be closed and linear operators with domain $D(A) \cap D(E) \neq \{0\}$ defined on a Banach space X. We say that the pair (A, E) is the generator of a* Sobolev type resolvent family, *if there exist $M > 0$ and $\omega \geq 0$ and a strongly continuous function $S : [0, \infty) \to \mathcal{B}([D(E)], X)$ such* *that* $\|S(t)\| \leq Me^{\omega t}$ *for all* $t \geq 0$, $\{\lambda : \mathrm{Re}\lambda > \omega\} \subset \rho_E(A)$ *and for all* $x \in D(E)$,

$$(\lambda E - A)^{-1}Ex = \int_0^\infty e^{-\lambda t} S(t)x dt, \quad \mathrm{Re}\lambda > \omega,$$

where $\rho_E(A)$ *denotes the set* $\rho_E(A) := \{\mu \in \mathbb{C} : (\mu E - A)^{-1}$*is invertible and* $(\mu E - A)^{-1}E$ *is bounded*$\}$. *In this case,* $\{S(t)\}_{t\geq 0}$ *is called the* Sobolev type resolvent family *generated by the pair* (A, E).

For $\alpha > 0$, g_α defines the function $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\Gamma(\cdot)$ is the gamma function. We note that if $\alpha, \beta > 0$, then the semigroup property holds: $g_{\alpha+\beta} = g_\alpha * g_\beta$, where $(f * g)$ denotes the usual finite convolution $(f * g)(t) = \int_0^t f(t-s)g(s)ds$.

The Riemann-Liouville fractional integral of order $\alpha > 0$ of a vector-valued function $f : [0, \infty) \to X$ is defined by

$$I^\alpha f(t) := (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s)f(s)ds.$$

The Caputo and Riemann-Liouville fractional derivatives of order $\alpha > 0$ of f are, respectively, defined by

$$D_t^\alpha f(t) := (g_{m-\alpha} * f^{(m)})(t) = \int_0^t g_{m-\alpha}(t-s)f^{(m)}(s)ds,$$

and

$$D^\alpha f(t) := \frac{d^m}{dt^m}\int_0^t g_{m-\alpha}(t-s)f(s)ds,$$

where $m = \lceil\alpha\rceil$ is the smallest integer greater than or equal to α. We notice that if $\alpha = m \in \mathbb{N}$, then $D_t^m = D^m = \frac{d^m}{dt^m}$. We refer to the reader to [28, 31] for further details, examples and applications on fractional calculus.

For a locally integrable function $f : [0, \infty) \to X$, we denote by $\hat{f}(\lambda)$ (or $\mathcal{L}(f)(\lambda)$) the *Laplace transform* of f :

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt,$$

provided the integral converges for some $\lambda \in \mathbb{C}$. Applying the properties of the Laplace transform, an easy computation shows that

$$\left\{\begin{array}{lcl} \widehat{D_t^\alpha f}(\lambda) & = & \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1}f(0) \qquad \text{and} \\ \widehat{D^\alpha f}(\lambda) & = & \lambda^\alpha \hat{f}(\lambda) - (g_{1-\alpha} * f)(0), \end{array}\right. \tag{2.1}$$

for $0 < \alpha \leq 1$, and

$$\left\{\begin{array}{lcl} \widehat{D_t^\alpha f}(\lambda) & = & \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1}f(0) - \lambda^{\alpha-2}f'(0) \quad \text{and} \\ \widehat{D^\alpha f}(\lambda) & = & \lambda^\alpha \hat{f}(\lambda) - \lambda(g_{2-\alpha} * f)(0) - (g_{2-\alpha} * f)'(0), \end{array}\right. \tag{2.2}$$

for $1 < \alpha \leq 2$. Here, the power λ^α is uniquely defined by $\lambda^\alpha := |\lambda|^\alpha e^{i\arg(\lambda)}$, with $-\pi < \arg(\lambda) < \pi$.

Now, we recall two important functions in fractional calculus. For $\alpha, \beta > 0$ and $z \in \mathbb{C}$, the Mittag-Leffler function $E_{\alpha,\beta}$ and its Laplace transform $\mathcal{L}$ are defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \mathcal{L}(t^{\beta-1}E_{\alpha,\beta}(\rho t^{\alpha}))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - \rho},$$

$\rho \in \mathbb{C}$, $\mathrm{Re}\lambda > |\rho|^{1/\alpha}$. For $\alpha > -1, \beta \in \mathbb{C}$ and $z \in \mathbb{C}$, the Wright function $W_{\alpha,\beta}$ is defined by

$$W_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)}.$$

If $\beta \geq 0$, then it is easy to prove (see [28]) that

$$W_{\alpha,\beta}(z) := \frac{1}{2\pi i} \int_{Ha} \mu^{-\beta} e^{\mu + z\mu^{-\alpha}} d\mu,$$

for all $z \in \mathbb{C}$ and $\alpha > -1$, where Ha denotes the Hankel path defined as a contour that begins and $t = -\infty - ia$ $(a > 0)$, encircles the branch cut that lies along the negative real axis, and ends up at $t = -\infty + ib$ $(b > 0)$, see for instance [28].

Definition 2. [1, Definition 3.1] *For $0 < \alpha < 1$ and $\beta \geq 0$, we define the function $\psi_{\alpha,\beta}$ in two variables by*

$$\psi_{\alpha,\beta}(t,s) := t^{\beta-1} W_{-\alpha,\beta}(-st^{\alpha}), \quad t > 0, \ s \in \mathbb{C}.$$

By [1, Theorem 3.2] it follows that if $0 < \alpha < 1$ and $\beta \geq 0$, then $\psi_{\alpha,\beta}(t,s) \geq 0$ for $t, s > 0$ and that

$$\int_0^{\infty} e^{-\lambda t} \psi_{\alpha,\beta}(t,s) dt = \lambda^{-\beta} e^{-\lambda^{\alpha} s}, \quad \text{for} \quad s, \lambda > 0. \tag{2.3}$$

Moreover, if $0 < \alpha < 1$, $\beta \geq 0$ and $\delta > 0$, then (see [1, Theorem 3.2])

$$\int_0^{\infty} g_{\delta}(s) \psi_{\alpha,\beta}(t,s) dt = g_{\alpha\delta+\beta}(t), \text{ for } t > 0, \tag{2.4}$$

and

$$\int_0^{\infty} e^{\lambda s} \psi_{\alpha,\beta}(t,s) ds = t^{\alpha+\beta-1} E_{\alpha,\alpha+\beta}(\lambda t^{\alpha}), \text{ for } t > 0 \text{ and } \lambda \in \mathbb{C}. \tag{2.5}$$

Definition 3. *Let $A : D(A) \subseteq X \to X$, $E : D(E) \subseteq X \to X$ be closed linear operators defined on a Banach space X satisfying $D(A) \cap D(E) \neq \{0\}$. Let $0 < \beta < \alpha$. We say that the pair (A, E) is the generator of an (α, β)-Sobolev type resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $S^E_{\alpha,\beta} : [0, \infty) \to \mathcal{B}([D(E)], X)$ such that $S^E_{\alpha,\beta}(t)$ is exponentially bounded, $\{\lambda^{\alpha} : \mathrm{Re}\lambda > \omega\} \subset \rho_E(A)$, and for all $x \in D(E)$,*

$$\lambda^{\alpha-\beta} \left(\lambda^{\alpha} E - A\right)^{-1} Ex = \int_0^{\infty} e^{-\lambda t} S^E_{\alpha,\beta}(t) x dt, \quad \mathrm{Re}\lambda > \omega. \tag{2.6}$$

In this case, $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ *is called the* (α,β)-Sobolev type resolvent family *generated by the pair* (A,E).

Lemma 4. *Suppose that* (A,E) *generates an* (α,β)*-Sobolev type resolvent family* $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$. *Then* $(g_\alpha * S^E_{\alpha,\beta})(t)x \in D(A)\cap D(E)$ *for all* $x\in D(E), t\geq 0$ *and*

$$ES^E_{\alpha,\beta}(t)x = g_\beta(t)Ex + A\int_0^t g_\alpha(t-s)S^E_{\alpha,\beta}(s)x ds. \tag{2.7}$$

Proof. Let $x\in D(E)$. For $t\geq 0$ we define $z(t) := (g_\alpha * S^E_{\alpha,\beta})(t)x$. Since $S^E_{\alpha,\beta}(t)$ is exponentially bounded we obtain that z is Laplace transformable and $\hat{z}(\lambda) = \frac{1}{\lambda^\alpha}\widehat{S^E_{\alpha,\beta}}(\lambda)x$. Now, if $\lambda^\alpha \in \rho_E(A)$ with Re$\lambda > \omega$ we have by (2.6)

$$\hat{z}(\lambda) = \frac{1}{\lambda^\alpha}\lambda^{\alpha-\beta}\left(\lambda^\alpha E - A\right)^{-1}Ex \in D(A)\cap D(E).$$

By [3, Proposition 1.7.6], $z(t)\in D(A)\cap D(E)$ for all $t\geq 0$. Finally, by (2.6) we obtain (2.7) by uniqueness of the Laplace transform. □

We notice that the Laplace transform of the operators Q and R defined in (1.3) satisfy (see [22, p. 513])

$$\hat{Q}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha E - A)^{-1}E \quad \text{and} \quad \hat{R}(\lambda) = (\lambda^\alpha E - A)^{-1}E,$$

for λ large enough, and therefore, Q and R are respectively, an $(\alpha,1)$ and an (α,α)-Sobolev type resolvent family.

This notion of Sobolev type resolvent family corresponds to an extension of the concept of (g_α, g_β)-regularized families introduced in [26] in the case $E = I$ (where I denotes the identity operator defined in X), and therefore, an extension of the concepts of C_0-semigroups, cosine families, integrated semigroups, among others, see for instance [3]. It is a well known fact that the C_0-semigroups of linear operators are an important tool in the study of mild solution to abstract first order differential equations in Banach spaces. On the other hand, we observe that if $\alpha=\beta=1$ in Definition 3, then the $(1,1)$-Sobolev type resolvent family corresponds to notion of Sobolev type resolvent family given in Definition 1, which are also of crucial importance in the existence of mild solution to the degenerate Cauchy problem (1.4), see for instance [18] and the references therein. Now, we define the concept of *mild solution* to the problems (1.5)–(1.8). The concept of (α,β)-Sobolev type resolvent family will be crucial here.

We first consider the case $0<\alpha<1$. Suppose that the pair (A,E) is the generator of the $(\alpha,1)$-Sobolev type resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$. Then, the property of the Laplace transform for the Caputo fractional derivative (2.1) allows to define the mild solution to problem (1.5) as

$$u(t) \;=\; S^E_{\alpha,1}(t)u_0 + \int_0^t (g_{1-\alpha} * S^E_{\alpha,1})(t-s)f(s)ds, \tag{2.8}$$

Similarly, if now (A, E) is the generator of the Sobolev type resolvent family $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$, then by the property (2.1), we can define the mild solution to problem (1.6) as

$$u(t) = S^E_{\alpha,\alpha}(t)u_0 + \int_0^t S^E_{\alpha,\alpha}(t-s)f(s)ds. \tag{2.9}$$

Now, we consider the case $1 < \alpha < 2$. If (A, E) is the generator of the Sobolev type resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$, then by using (2.2) we define the mild solution to (1.7) as

$$u(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)f(s)ds, \tag{2.10}$$

and if (A, E) generates the Sobolev type resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$, then the property (2.2) for the Riemann-Liouville fractional derivative allows to define the solution to (1.8) as

$$u(t) = S^E_{\alpha,\alpha-1}(t)u_0+(g_1*S^E_{\alpha,\alpha-1})(t)u_1+\int_0^t (g_1*S^E_{\alpha,\alpha-1})(t-s)f(s)ds. \tag{2.11}$$

3. Subordination principle for Sobolev type resolvent families

In order to define the mild solutions (2.8)–(2.11) we need to ensure the existence of the families $\{S^E_{\alpha,\beta}(t)\}_{t\geq 0}$ for suitable positive α and β. In this section, we prove a subordination principle which will be of a crucial importance to prove the existence of mild solutions to problems (1.5)–(1.8).

Theorem 1 (Subordination)**.** *Let X be a Banach space. Take $0 < \mu \leq 2$ and $\nu > 0$, and assume that the pair (A, E) generates a (μ, ν)-Sobolev type resolvent family $\{S^E_{\mu,\nu}(t)\}_{t\geq 0}$ defined in X, such that $\|S^E_{\mu,\nu}(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, where $M, \omega \geq 0$. Take $0 < \delta < 1$ and $\varepsilon \geq 0$. Then, the pair (A, E) is the generator of the $(\delta\mu, \delta\nu+\varepsilon)$-Sobolev type resolvent family $\{S^E_{\delta\mu,\delta\nu+\varepsilon}(t)\}_{t\geq 0}$ defined by*

$$S^E_{\delta\mu,\delta\nu+\varepsilon}(t)x := \int_0^\infty \psi_{\delta,\varepsilon}(t, s)S^E_{\mu,\nu}(s)xds, \quad t \geq 0, x \in X \tag{3.1}$$

where $\psi_{\delta,\varepsilon}$ is the Wright type function given in Definition 2. Moreover, if $\varepsilon > 0$, then

$$S^E_{\delta\mu,\delta\nu+\varepsilon}(t)x = (g_\varepsilon * S^E_{\delta\mu,\delta\nu})(t)x, \tag{3.2}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows similarly to [1, Theorem 4.5]. We give here the details for the sake of completeness. We need to prove that the family $\{S^E_{\delta\mu,\delta\nu+\varepsilon}(t)\}_{t\geq 0}$ defined by (3.1) defines a $(\delta\mu, \delta\nu+\varepsilon)$-Sobolev type resolvent family. Since $\|S^E_{\mu,\nu}(t)\| \leq Me^{\omega t}$ we obtain that $S^E_{\delta\mu,\delta\nu+\varepsilon}(t)$ is exponentially bounded. In fact, since $0 < \delta < 1$ and $\varepsilon \geq 0$ we obtain $\psi_{\delta,\varepsilon}(t,s) \geq 0$ and if $x \in X$, by (3.1) and (2.5) we have

$$\begin{aligned}\|S^E_{\delta\mu,\delta\nu+\varepsilon}(t)x\| &\leq \int_0^\infty \psi_{\delta,\varepsilon}(t,s)\|S^E_{\mu,\nu}(s)x\|ds \\ &\leq M\int_0^\infty \psi_{\delta,\varepsilon}(t,s)e^{\omega s}\|x\|ds = Mt^{\delta+\varepsilon-1}E_{\delta,\delta+\varepsilon}(\omega t^\delta).\end{aligned}$$

By [4, Formula (1.27)], there exists a constant $C > 0$ such that $E_{\alpha,\beta}(\lambda t^\alpha) \leq C\lambda^{\frac{1-\beta}{\alpha}} t^{1-\beta} e^{\lambda^{\frac{1}{\alpha}} t}$, for all $0 < \alpha < 2$, $\beta > 0$, $\lambda \geq 0$ and $t \geq 0$, which implies the existence of a constant $K > 0$ such that

$$\|S^E_{\delta\mu,\delta\nu+\varepsilon}(t)x\| \leq Mt^{\delta+\varepsilon-1}E_{\delta,\delta+\varepsilon}(\omega t^\delta) \leq Ke^{\omega^{\frac{1}{\delta}} t}.$$

This shows that $\{S^E_{\delta\mu,\delta\nu+\varepsilon}(t)\}_{t\geq 0}$ is exponentially bounded. In order to prove that $\{S^E_{\delta\mu,\delta\nu+\varepsilon}(t)\}_{t\geq 0}$ is strongly continuous, we first notice that $S^E_{\delta\mu,\delta\nu+\varepsilon}(t)$ is strongly continuous for all $t > 0$ by its definition given in (3.1) and the strong continuity of $\{S^E_{\mu,\nu}(t)\}_{t\geq 0}$. Now, to prove the strong continuity at the origin we notice that for all $x \in X$ we have by (2.4)

$$\begin{aligned}\frac{\|S^E_{\delta\mu,\delta\nu+\varepsilon}(t)x - g_{\delta\nu+\varepsilon}(t)x\|}{g_{\delta\nu+\varepsilon}(t)} &\leq \frac{1}{g_{\delta\nu+\varepsilon}(t)}\int_0^\infty \psi_{\delta,\varepsilon}(t,s)\|S^E_{\mu,\nu}(s)x - g_\nu(s)x\|ds \\ &= \Gamma(\delta\nu+\varepsilon)\int_0^\infty t^{\delta-\delta\nu}W_{-\delta,\varepsilon}(-r)\|S^E_{\mu,\nu}(rt^\delta)x - g_\nu(rt^\delta)x\|dr \\ &= \Gamma(\delta\nu+\varepsilon)\int_0^\infty \frac{g_\nu(r)W_{-\delta,\varepsilon}(-r)}{g_\nu(rt^\delta)}\|S^E_{\mu,\nu}(rt^\delta)x - g_\nu(rt^\delta)x\|dr.\end{aligned}$$

Since $\{S^E_{\mu,\nu}(t)\}_{t\geq 0}$ is a (μ,ν)-Sobolev type resolvent family, we obtain, by the dominated convergence theorem, that

$$\frac{\|S^E_{\delta\mu,\delta\nu+\varepsilon}(t)x - g_{\delta\nu+\varepsilon}(t)x\|}{g_{\delta\nu+\varepsilon}(t)} \to 0, \quad \text{as} \quad t \to 0^+.$$

On the other hand, since $\{S^E_{\mu,\nu}(t)\}_{t\geq 0}$ is a (μ,ν)-Sobolev type resolvent family we have by (2.3) and Fubini's theorem that

$$\begin{aligned}\int_0^\infty e^{-\lambda t}S^E_{\delta\mu,\delta\nu+\varepsilon}(t)xdt &= \int_0^\infty e^{-\lambda t}\left(\int_0^\infty \psi_{\delta,\varepsilon}(t,s)S^E_{\mu,\nu}(s)xds\right)dt \\ &= \int_0^\infty\left(\int_0^\infty e^{-\lambda t}\psi_{\delta,\varepsilon}(t,s)dt\right)S^E_{\mu,\nu}(s)xds \\ &= \int_0^\infty \lambda^{-\varepsilon}e^{-\lambda^\delta s}S^E_{\mu,\nu}(s)xds \\ &= \lambda^{\delta\mu-(\delta\nu+\varepsilon)}(\lambda^{\delta\mu}E - A)^{-1}Ex,\end{aligned}$$

for all $\lambda > \omega^{1/\delta}$ such that $\lambda^{\delta\mu} \in \rho_E(A)$. The identity (3.2) follows directly from the properties of the Laplace transform. This concludes the proof of the theorem. $\square$

Corollary 2. *Let X be a Banach space. Assume that the pair (A,E) generates a Sobolev type resolvent family $\{S^E(t)\}_{t\geq 0}$ defined in X. If $0<\alpha<1$, then the pair (A,E) is the generator of the $(\alpha,1)$-Sobolev type resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ defined by*

$$S^E_{\alpha,1}(t)x := \int_0^\infty \psi_{\alpha,1-\alpha}(t,s)S^E(s)xds, \quad t\geq 0,\ x\in X, \tag{3.3}$$

where $\psi_{\alpha,1-\alpha}$ is the Wright type function given by

$$\psi_{\alpha,1-\alpha}(t,s) = \frac{1}{\pi}\int_0^\infty \rho^{\alpha-1}e^{-s\rho^\alpha\cos\alpha(\pi-\theta)-t\rho\cos\theta} \times \sin\left(t\rho\sin\theta - s\rho^\alpha\sin\alpha(\pi-\theta)+\alpha(\pi-\theta)\right)d\rho, \tag{3.4}$$

for $\theta\in(\pi-\frac{\pi}{2\alpha},\pi/2)$.

Proof. If $\mu=\nu=1$ and $\varepsilon=1-\alpha$ in the subordination Theorem 1 we obtain (3.3). On the other hand, by the identity (2.3) we have

$$\int_0^\infty e^{-\lambda t}\psi_{\alpha,1-\alpha}(t,s)dt = \lambda^{\alpha-1}e^{-\lambda^\alpha s}, \quad \lambda,\ s>0,$$

and by [21, Corollary 3.3 (b)], formula (3.4) holds as a consequence of the uniqueness of the Laplace transform. □

Corollary 3. *Let X be a Banach space. Assume that the pair (A,E) generates a Sobolev type resolvent family $\{S^E(t)\}_{t\geq 0}$ defined in X. If $0<\alpha<1$, then the pair (A,E) is the generator of the (α,α)-Sobolev type resolvent family $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ defined by*

$$S^E_{\alpha,\alpha}(t)x := \int_0^\infty \psi_{\alpha,0}(t,s)S^E(s)xds, \quad t\geq 0,\ x\in X, \tag{3.5}$$

where $\psi_{\alpha,0}$ is the Wright type function given by

$$\psi_{\alpha,0}(t,s) = \frac{1}{\pi}\int_0^\infty e^{t\rho\cos\theta - s\rho^\alpha\cos\alpha\theta}\cdot\sin(t\rho\sin\theta - s\rho\sin\alpha\theta+\theta)d\rho, \tag{3.6}$$

for $\pi/2<\theta<\pi$.

Proof. By the subordination Theorem 1 we obtain (3.5) as a consequence of (3.1), with $\mu=\nu=1$ and $\varepsilon=0$. The identity (2.3) implies that

$$\int_0^\infty e^{-\lambda t}\psi_{\alpha,0}(t,s)dt = e^{-\lambda^\alpha s}, \quad \lambda,\ s>0,$$

and by [21, Formula (3.8) in Corollary 3.3], we obtain (3.6) as a consequence of the uniqueness of the Laplace transform. □

Corollary 4. *Let X be a Banach space. Assume that the pair (A,E) generates a $(2,1)$-Sobolev type resolvent family $\{S^E_{2,1}(t)\}_{t\geq 0}$ defined in X. If $1<\alpha<2$, then the pair (A,E) is the generator of the $(\alpha,1)$-Sobolev type resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ defined by*

$$S^E_{\alpha,1}(t)x := \int_0^\infty \psi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s)S^E_{2,1}(s)xds, \quad t\geq 0, x\in X, \tag{3.7}$$

where $\psi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}$ *is the Wright type function given by*

$$\psi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s) = \frac{1}{\pi}\int_0^\infty \rho^{\frac{\alpha}{2}-1} e^{-s\rho^{\frac{\alpha}{2}}\cos\frac{\alpha}{2}(\pi-\theta)-t\rho\cos\theta}$$

$$\times \sin\left(t\rho\sin\theta - s\rho^{\frac{\alpha}{2}}\sin\tfrac{\alpha}{2}(\pi-\theta)+\tfrac{\alpha}{2}(\pi-\theta)\right)d\rho, \tag{3.8}$$

for $\theta \in (\pi - \frac{2}{\alpha}, \pi/2)$.

Proof. We take $\mu = 2, \nu = 1, \delta = \frac{\alpha}{2}$ and $\varepsilon = 1 - \frac{\alpha}{2}$ in Theorem 1 to obtain the formula (3.7). On the other hand, by (2.3) we have

$$\int_0^\infty e^{-\lambda t}\psi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s)dt = \lambda^{\frac{\alpha}{2}-1}e^{-\lambda^{\frac{\alpha}{2}}s}, \quad \lambda, s > 0,$$

and by [21, Corollary 3.3 (b)], we obtain (3.8). □

Corollary 5. *Let* X *be a Banach space. Assume that the pair* (A,E) *generates a* $(2,1)$*-Sobolev type resolvent family* $\{S^E_{2,1}(t)\}_{t\geq 0}$ *defined in* X*. If* $1 < \alpha < 2$*, then the pair* (A,E) *is the generator of the* (α,α)*-Sobolev type resolvent family* $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ *defined by*

$$S^E_{\alpha,\alpha}(t)x := \int_0^\infty \psi_{\frac{\alpha}{2},\frac{\alpha}{2}}(t,s)S^E_{2,1}(s)xds, \quad t \geq 0, x \in X, \tag{3.9}$$

where $\psi_{\frac{\alpha}{2},\frac{\alpha}{2}}$ *is the Wright type function given by*

$$\psi_{\frac{\alpha}{2},\frac{\alpha}{2}}(t,s) = (g_{\frac{\alpha}{2}} * \psi_{\frac{\alpha}{2},0}(\cdot,s))(t), \tag{3.10}$$

where $\psi_{\frac{\alpha}{2},0}(\cdot,s)$ *is given in* (3.6).

Proof. If we take $\mu = 2, \nu = 1, \delta = \frac{\alpha}{2}$ and $\varepsilon = \frac{\alpha}{2}$ in Theorem 1, then we obtain the formula (3.9). On the other hand, by (2.3) we can write

$$\int_0^\infty e^{-\lambda t}\psi_{\frac{\alpha}{2},\frac{\alpha}{2}}(t,s)dt = \lambda^{-\frac{\alpha}{2}}e^{-\lambda^{\frac{\alpha}{2}}s} = \frac{1}{\lambda^{\frac{\alpha}{2}}}e^{-\lambda^{\frac{\alpha}{2}}s}, \quad \lambda, s > 0.$$

Since $\hat{g}_{\frac{\alpha}{2}}(\lambda) = \frac{1}{\lambda^{\frac{\alpha}{2}}}$ for $\lambda > 0$, we obtain (3.10) as a consequence of the uniqueness of the Laplace transform and (3.6). □

4. Existence of mild solutions to fractional diffusion equations of Sobolev type

In this section we study the existence of mild solution to problems (1.5)–(1.8) for the Caputo and Riemann-Liouville fractional derivatives of order $0 < \alpha < 1$ and $1 < \alpha < 2$. We remark here that, to the best of our knowledge, the case $1 < \alpha < 2$ seems to be new in the existing literature.

We first discuss the case $0 < \alpha < 1$. We recall that a Sobolev type resolvent family corresponds to a $(1,1)$-Sobolev type resolvent family.

Theorem 1. *Let* X *be a Banach space. Assume that the pair* (A,E) *generates a Sobolev type resolvent family* $\{S^E(t)\}_{t\geq 0}$ *defined in* X*. If* $0 < \alpha < 1$ *and* f *is Laplace transformable, then the problem* (1.5) *for the Caputo fractional derivative has a unique mild solution given by*

$$u(t) = S^E_{\alpha,1}(t)u_0 + \int_0^t (g_{1-\alpha} * S^E_{\alpha,1})(t-s)f(s)ds,$$

where $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ *is given in* (3.3).

Proof. Since the pair (A, E) is the generator of the Sobolev type resolvent family $\{S^E(t)\}_{t\geq 0}$, Corollary 2 implies that the pair (A, E) is the generator of the $(\alpha, 1)$-Sobolev type resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ given by (3.3) and the solution is obtained as a consequence of the uniqueness of the Laplace transform. □

Assume that (A, E) generates a Sobolev type resolvent family $\{S^E(t)\}_{t\geq 0}$. If E is the identity operator in the Banach space X, then A generates a C_0-semigroup, and therefore, the next result is an extension of [1, Theorem 5.1].

Theorem 2. *Let X be a Banach space. Assume that the pair (A, E) generates a Sobolev type resolvent family $\{S^E(t)\}_{t\geq 0}$ defined in X. If $0 < \alpha < 1$ and f is Laplace transformable, then the problem* (1.6) *for the Riemann-Liouville fractional derivative has a unique mild solution given by*

$$u(t) = S^E_{\alpha,\alpha}(t)u_0 + \int_0^t S^E_{\alpha,\alpha}(t-s)f(s)ds,$$

where $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ *is given in* (3.5).

Proof. It follows similarly to the proof of Theorem 1. □

Now we consider the case $1 < \alpha < 2$. If (A, E) generates a $(2, 1)$-Sobolev type resolvent family $\{S^E(t)\}_{t\geq 0}$ and E is the identity operator in the Banach space X, then $\{S^I(t)\}_{t\geq 0}$ corresponds to a cosine family generated by A (see [3]). We have the following results.

Theorem 3. *Let X be a Banach space. Assume that the pair (A, E) generates a $(2, 1)$-Sobolev type resolvent family $\{S^E_{2,1}(t)\}_{t\geq 0}$ defined in X. If $1 < \alpha < 2$ and f is Laplace transformable, then the problem* (1.7) *for the Caputo fractional derivative has a unique mild solution given by*

$$u(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)f(s)ds,$$

where $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ *is given in* (3.7).

Proof. Corollary 4 implies that the pair (A, E) generates the $(\alpha, 1)$-Sobolev type resolvent family $\{S_{\alpha,1}(t)\}_{t\geq 0}$ given by (3.7). The result holds by the uniqueness of the Laplace transform. □

Theorem 4. *Let X be a Banach space. Assume that the pair (A, E) generates a $(2, 1)$-Sobolev type resolvent family $\{S^E_{2,1}(t)\}_{t\geq 0}$ defined in X. If $1 < \alpha < 2$ and f is Laplace transformable, then the problem*

$$\left\{ \begin{array}{rcl} D^\alpha(Eu)(t) &=& Au(t) + Ef(t), t \geq 0, \\ (g_{2-\alpha} * Eu)(0) &=& 0, \\ (g_{2-\alpha} * Eu)'(0) &=& Eu_1 \end{array} \right. \tag{4.1}$$

for the Riemann-Liouville fractional derivative has a unique mild solution given by

$$u(t) = S^E_{\alpha,\alpha}(t)u_1 + \int_0^t S^E_{\alpha,\alpha}(t-s)f(s)ds. \tag{4.2}$$

where $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ *is given in* (3.9).

Proof. Since $(g_{2-\alpha} * Eu)(0) = 0$ the mild solution given in (2.11) reads

$$u(t) = (g_1 * S^E_{\alpha,\alpha-1})(t)u_1 + \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)f(s)ds,$$

which, by the uniqueness of the Laplace transform, can be written as (4.2). Since (A, E) generates a $(2,1)$-Sobolev type resolvent family, the Corollary 5 asserts that that the pair (A, E) generates the (α, α)-Sobolev type resolvent family $\{S_{\alpha,\alpha}(t)\}_{t\geq 0}$ given by (3.9) and the result holds by the uniqueness of the Laplace transform. □

We remark that the problem (4.1) with the first initial condition equal to zero has been widely studied in the last years, see for instance [23, 30] and the references therein. If $(g_{2-\alpha} * Eu)(0) \neq 0$ in (4.1), then the mild solution to (1.8) is given by

$$u(t) = S^E_{\alpha,\alpha-1}(t)u_0 + (g_1 * S^E_{\alpha,\alpha-1})(t)u_1 + \int_0^t (g_1 * S^E_{\alpha,\alpha-1})(t-s)f(s)ds.$$

However, the Subordination Theorem 1 can not be used to obtain the Sobolev type resolvent family $\{S^E_{\alpha,\alpha-1}(t)\}_{t\geq 0}$ by assuming that the pair (A, E) generates a $(2,1)$ or a $(2,2)$-Sobolev type resolvent family. If fact, if $\mu = 2, \nu = 1$ and $\delta = \frac{\alpha}{2}$ in Theorem 1, then $\delta\nu + \varepsilon = \alpha - 1$ is equivalent to say that $\varepsilon = \frac{\alpha}{2} - 1 < 0$. Therefore, (A, E) can not be the generator of a $(2,1)$-Sobolev type resolvent family in order to ensure that (A, E) is the generator of an $(\alpha, \alpha-1)$-Sobolev type resolvent family. Similarly, if $\mu = 2, \nu = 2$ and $\delta = \frac{\alpha}{2}$ in Theorem 1, we obtain $\varepsilon = -1 < 0$, which is impossible and we conclude that (A, E) can not be the generator of a $(2,2)$-Sobolev type resolvent family which is also the generator of an $(\alpha, \alpha-1)$-Sobolev type resolvent family.

5. Applications

Let $0 < \alpha < 1$. We consider the following fractional differential equations of Sobolev type

(5.1)
$$\begin{cases} D^\alpha_t \left(\sum_{|p|\leq 2m} b_p D^p_x u \right)(t,x) &= \sum_{|p|\leq 2m} a_p D^p_x u(t,x) + \sum_{|p|\leq 2m} b_p D^p_x f(t,x), \\ \sum_{|p|\leq 2m} b_p D^p_x u(0,x) &= \sum_{|p|\leq 2m} b_p D^p_x u_0(x), \end{cases}$$

where $t \geq 0$, $x \in \mathbb{R}^n$, the function u_0 is a fixed complex valued function defined in $\mathbb{R}^n$, p denotes the n-dimensional multi-index $p = (p_1, ..., p_n) \in \mathbb{N}_0^n := \mathbb{N}_0 \times ... \times \mathbb{N}_0$, $|p| = p_1 + ... + p_n$ and

$$D_x^p = D_{x_1}^{p_1} ... D_{x_n}^{p_n} = \left(\frac{\partial}{\partial x_1}\right)^{p_1} ... \left(\frac{\partial}{\partial x_n}\right)^{p_n},$$

and moreover, m is a fixed positive integer, $a_p \in \mathbb{C}$ for each multi-index $p \in \mathbb{N}_0^n$ with $|p| \leq m$, and $b_p \in \mathbb{C}$ for each $p \in \mathbb{N}_0^n$, with $|p| \leq m$.

Given a $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz's space of all rapidly decreasing functions on $\mathbb{R}^n$, the Fourier transform and its inverse transform are denoted, respectively, by

$$(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^n} e^{-i\langle \eta, \xi \rangle} f(\eta) d\eta \text{ and } (\mathcal{F}^{-1}f)(\eta) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle \eta, \xi \rangle} f(\xi) d\xi.$$

Now, let $X = L^2(\mathbb{R}^n)$ be the Hilbert space of all square integrable functions define on $\mathbb{R}^n$ and define the operators A and E by

$$Au := \sum_{|p| \leq 2m} a_p D_x^p u, \text{ with domian } D(A) := \left\{ u \in X : \sum_{|p| \leq 2m} a_p D_x^p u \in X \right\}$$

and

$$Eu := \sum_{|p| \leq 2m} b_p D_x^p u, \text{ with domian } D(E) := \left\{ u \in X : \sum_{|p| \leq 2m} b_p D_x^p u \in X \right\}$$

Clearly, A and E are closed linear operators. For $\xi \in \mathbb{R}^n$, the symbol of A and E will be denoted respectively by

$$a(\xi) := \sum_{|p| \leq 2m} i^{|p|} a_p \xi^p, \quad \text{and} \quad b(\xi) := \sum_{|p| \leq 2m} i^{|p|} b_p \xi^p.$$

In these conditions, the problem (5.1) for the Caputo fractional derivative can be written in the abstract form (1.5). Similarly, if we consider the problem for the Riemann-Liouville fractional derivative

(5.2)

$$\begin{cases} D^\alpha \left(\sum_{|p| \leq 2m} b_p D_x^p u \right)(t, x) = \sum_{|p| \leq 2m} a_p D_x^p u(t, x) + \sum_{|p| \leq 2m} b_p D_x^p f(t, x), \\ \sum_{|p| \leq 2m} \left(g_{1-\alpha} * b_p D_x^p u \right)(0, x) = \sum_{|p| \leq 2m} b_p D_x^p u_0(x), \end{cases}$$

where $t \geq 0$, then it can be written in the abstract form (1.6) with the same operators A and E. Now, we recall the following result in [24].

Theorem 1. [24, Theorem 2.2] *Let $X = L^2(\mathbb{R}^n)$. Assume that $b(\xi) \neq 0$ for each $\xi \in \mathbb{R}^n$ and $\omega := \sup_{\xi \in \mathbb{R}^n} \operatorname{Re}\left(a(\xi) b^{-1}(\xi)\right) \leq 0$. Then, the pair (A, E) is* *the generator of a Sobolev type resolvent family* $\{S^E(t)\}_{t\geq 0}$, *and there exists a positive constant* C *such that*

$$\|S^E(t)\| \leq Ce^{\omega t}, \quad t \geq 0.$$

Since (A, E) generates a Sobolev type resolvent family $\{S^E(t)\}_{t\geq 0}$, Corollaries 2 and 3 imply the following results for $0 < \alpha < 1$.

Proposition 2. *Let* $X = L^2(\mathbb{R}^n)$ *and* $0 < \alpha < 1$. *Assume that* $b(\xi) \neq 0$ *for each* $\xi \in \mathbb{R}^n$ *and* $\omega := \sup_{\xi\in\mathbb{R}^n} \mathrm{Re}\left(a(\xi)b^{-1}(\xi)\right) \leq 0$. *Then, the pair* (A, E) *is the generator of the* $(\alpha, 1)$*-Sobolev type resolvent family* $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ *given by* (3.3).

Proposition 3. *Let* $X = L^2(\mathbb{R}^n)$ *and* $0 < \alpha < 1$. *Assume that* $b(\xi) \neq 0$ *for each* $\xi \in \mathbb{R}^n$ *and* $\omega := \sup_{\xi\in\mathbb{R}^n} \mathrm{Re}\left(a(\xi)b^{-1}(\xi)\right) \leq 0$. *Then, the pair* (A, E) *is the generator of the* (α, α)*-Sobolev type resolvent family* $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ *given by* (3.5).

As a consequence of Propositions 2 and 3 we have the following results for the existence of mild solutions in the case $0 < \alpha < 1$. Here A and E are defined as before.

Theorem 4. *Let* $0 < \alpha < 1$. *If* $f(t) \in D(E)$, *then the problem* (5.1) *for the Caputo fractional derivative has a unique mild solution given by*

$$u(t) = S^E_{\alpha,1}(t)u_0 + \int_0^t (g_{1-\alpha} * S^E_{\alpha,1})(t-s)f(s)ds,$$

where $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ *is given in Proposition 2.*

Proof. By Theorem 1, (A, E) is the generator of a Sobolev type resolvent family $\{S^E(t)\}_{t\geq 0}$, and by Proposition 2 (A, E) is also the generator of the $(\alpha, 1)$-Sobolev type resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ given by (3.3) and then, the result holds by Theorem 1. □

Theorem 5. *Let* $0 < \alpha < 1$. *If* $f(t) \in D(E)$, *then the problem* (5.2) *for the Riemann-Liouville fractional derivative has a unique mild solution given by*

$$u(t) = S^E_{\alpha,\alpha}(t)u_0 + \int_0^t S^E_{\alpha,\alpha}(t-s)f(s)ds,$$

where $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ *is given in Proposition 3.*

Proof. The proof follows similarly to the Proof of Theorem 4, by using Proposition 3 and Theorem 2. □

Now, we study the existence of mild solutions in case $1 < \alpha < 2$. First, we need to study the existence of mild solutions to the second order problem. Let $A : D(A) \subseteq X \to X$, $E : D(E) \subseteq X \to X$ be closed linear operators defined in a Banach space X satisfying $D(A) \cap D(E) \neq \{0\}$. Given $u_0, u_1 \in D(E)$ consider the second order problem of Sobolev type

$$\left\{\begin{array}{rcl} (Eu)''(t) &=& Au(t), \quad t \geq 0, \\ (Eu)(0) &=& Eu_0, \\ (Eu)'(0) &=& Eu_1 \end{array}\right. \tag{5.3}$$

A *classical solution* to (5.3) is a twice differentiable function $u : \mathbb{R}_+ \to X$ such that $u(t) \in D(A) \cap D(E)$ for each $t \geq 0$ and (5.3) holds for all $t \geq 0$. A *mild solution* to (5.3) is a function $u : \mathbb{R}_+ \to X$ such that $u(t) \in D(E)$ and $(g_2 * u)(t) \in D(A)$ for all $t \geq 0$, and

$$Eu(t) = Eu_0 + tEu_1 + A(g_2 * u)(t), \quad t \geq 0. \tag{5.4}$$

We notice that if u is a classical solution, then integrating (5.3) twice, we obtain that u is a mild solution. Moreover, if u is a twice differentiable function which is a mild solution to (5.3), then u is a classical solution.

Lemma 6. *Let $A : D(A) \subseteq X \to X$, $E : D(E) \subseteq X \to X$ be closed linear operators defined on a Banach space X satisfying $D(A) \cap D(E) \neq \{0\}$. Let u a continuous such that $\|u(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, where $\omega > 0$. Then u is a mild solution to* (5.3) *if and only if for all $\lambda > \omega$ we have*

$$\hat{u}(\lambda) \in D(A) \cap D(E) \qquad \textit{and} \qquad \lambda Eu_0 + Eu_1 = (\lambda^2 E - A)\hat{u}(\lambda). \tag{5.5}$$

Proof. The proof follows the same lines that [3, Proposition 3.14.1]. Let u be a mild solution to (5.3). Since u is Laplace transformable and A and E are closed operators, by [3, Proposition 1.7.6] we have $\hat{u}(\lambda) \in D(E)$ and moreover $\frac{\hat{u}(\lambda)}{\lambda^2} = \widehat{(g_2 * u)}(\lambda) \in D(A)$ by [3, Proposition 1.7.6], which implies that $\hat{u}(\lambda) \in D(A)$ for all $\lambda > \omega$. The equation in (5.5) follows taking Laplace transform in (5.4). Conversely, assume that (5.5) holds. By [3, Proposition 1.7.6] $u(t) \in D(E)$ for all $t \geq 0$. Define $v(t) := (g_1 * u)(t)$. Then v is Laplace transformable and

$$\int_0^\infty e^{-\lambda t}(g_1 * v)(t)dt = \frac{\hat{u}(\lambda)}{\lambda^2} = \widehat{(g_2 * u)}(\lambda) \in D(A).$$

The same Proposition 1.7.6 in [3] implies that $(g_2 * u)(t) = (g_1 * v)(t) \in D(A)$ for all $t \geq 0$. The equation in (5.5) implies (5.4) by the uniqueness of the Laplace transform. $\square$

Now, if $u : \mathbb{R}_+ \to X$ is continuous, then Lemma 6 shows that u is a mild solution to (5.3) if and only if $\hat{u}(\lambda) = \lambda(\lambda^2 E - A)^{-1}Eu_0 + (\lambda^2 E - A)^{-1}Eu_1$ for all $\lambda > \omega$. Assume that the pair (A, E) generates a $(2, 1)$-Sobolev type resolvent family $\{S^E_{2,1}(t)\}_{t\geq 0}$. Since $D^2_t = D^2 = \frac{d^2}{dt^2}$ we have that the second order problem of Sobolev type (5.3) has a unique mild solution (according to definition given in (2.10)) given by

$$u(t) \quad = \quad S^E_{2,1}(t)u_0 + (g_1 * S^E_{2,1})(t)u_1. \tag{5.6}$$

Since $\widehat{S^E_{2,1}}(\lambda) = \lambda(\lambda^2 E - A)^{-1}E$, then both definition of mild solution (5.4) and (5.6) to (5.3) are equivalent. Moreover, we have the following result.

Proposition 7. *Let $A : D(A) \subseteq X \to X$ be a closed linear operator and $E : D(E) \subseteq X \to X$ be a bounded linear operator defined in a Banach space X satisfying $D(A) \cap D(E) \neq \{0\}$. If for all initial value $x \in D(E)$ the* *problem*

$$\left\{ \begin{array}{rcl} (Eu)''(t) & = & Au(t), \quad t \geq 0, \\ (Eu)(0) & = & Ex, \\ (Eu)'(0) & = & 0 \end{array} \right. \tag{5.7}$$

has a unique mild solution u *with* $\|u(t)\| \leq Me^{\omega t}$*, where* $M, \omega > 0$*, then the pair* (A, E) *is the generator of a* $(2,1)$*-Sobolev type resolvent family.*

Proof. The proof follows similarly to [3, Theorem 3.1.12] and [34, Theorem 6.9]. Let $C(\mathbb{R}_+, X)$ be the space of all continuous functions $v : \mathbb{R}_+ \to X$. Let $u_x(t) := u(t, x)$ be the unique mild solution to (5.7). From the uniqueness, it follows that u is linear in x. Therefore, we can define for each $t \geq 0$ the linear operators $S^E_{2,1}(t) : X \to X$ by $S^E_{2,1}(t)x := u_x(t)$, $x \in X$. We claim that $S^E_{2,1}(t)$ is a bounded operator. In fact, the mapping $\varphi : X \to C(\mathbb{R}_+, X)$ given by $\varphi(x) = u_x$ is linear. Moreover, if $x_n \to x$ in X and $u_{x_n} \to u$ in $C(\mathbb{R}_+, X)$, then $(g_2 * u_{x_n})(t) \to (g_2 * u)(t)$ for all $t > 0$. By (5.4) we have $A(g_2 * u_{x_n})(t) = Eu_{x_n}(t) - Ex_n, \quad n \in \mathbb{N}$. Since A is closed and E is bounded, it follows that $(g_2 * u)(t) \in D(A)$ and $A(g_2 * u)(t) = \lim_{n\to\infty} Eu_{x_n}(t) - Ex_n = Eu(t) - Ex$, which means that u is a mild solution to (5.7) with the initial value x and therefore $\varphi(x) = u$. We conclude that φ has a closed graph, and by the closed graph theorem φ is continuous, which implies that $S^E_{2,1}(t)$ is strongly continuous.

On the other hand, since $\|u_x(t)\| \leq Me^{\omega t}$, we obtain $\|S^E_{2,1}(t)x\| \leq Me^{\omega t}$, which shows that $S^E_{2,1}(t)$ is exponentially bounded. Finally, by Lemma 6 it follows that $(\lambda^2 E - A)\hat{u}_x(\lambda) = \lambda Ex$ for all $\lambda > \omega$. Hence, if $\lambda^2 \in \rho_E(A)$, then $\widehat{S^E_{2,1}}(\lambda)x = \hat{u}_x(\lambda) = (\lambda^2 E - A)^{-1}Ex$. This proves the proposition. $\square$

Now, we take $1 < \alpha < 2$. Let Ω be a bounded open subset on $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let $T > 0$, $D = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \partial\Omega$. We consider

$$\left\{ \begin{array}{rcl} D^\alpha_t\,(m(x)u(t,x)) & = & \Delta u(t,x) + m(x)f(t,x), \quad t \geq 0, \\ m(x)u(0,x) & = & m(x)u_0(x), \\ (m(x)u)'(0,x) & = & m(x)v_0(x), \end{array} \right. \tag{5.8}$$

and

$$\left\{ \begin{array}{rcl} D^\alpha_t\,(m(x)u(t,x)) & = & \Delta u(t,x) + m(x)f(t,x), \quad t \geq 0, \\ (g_{2-\alpha} * m(x)u)(0,x) & = & 0, \\ (g_{2-\alpha} * m(x)u)'(0,x) & = & m(x)v_0(x), \end{array} \right. \tag{5.9}$$

where m is a positive and continuous function on $\overline{\Omega}$, u_0, v_0 is a complex valued function defined in Ω.

Let E be the multiplication operator by m. If we take $X = L^2(\Omega)$ and $A = \Delta$ with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ then by [18, Section 6.2], the problem

$$
(5.10) \qquad \left\{ \begin{array}{rcl} \frac{\partial^2}{\partial t^2}\left(m(x)u(t,x)\right) &=& \Delta u(t,x) + m(x)f(t,x), \quad t \geq 0, \\ m(x)u(0,x) &=& m(x)u_0(x), \\ (m(x)u)'(0,x) &=& 0, \end{array} \right.
$$

has a unique strict (and therefore mild) solution. We conclude by Proposition 7 that the pair (A,E) generates a $(2,1)$-Sobolev type resolvent family $\{S^E_{2,1}(t)\}_{t\geq 0}$. As a consequence of Corollaries 4 and 5 we have the following propositions.

Proposition 8. *Let $1 < \alpha < 2$. Let E be the multiplication operator by m, $X = L^2(\Omega)$ and $A = \Delta$ with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Then, the pair (A,E) is the generator of the $(\alpha,1)$-Sobolev type resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ given by*

$$
(5.11) \qquad S^E_{\alpha,1}(t)x := \int_0^\infty \psi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}(t,s)S^E_{2,1}(s)x\,ds, \quad t \geq 0, x \in X,
$$

where $\psi_{\frac{\alpha}{2},1-\frac{\alpha}{2}}$ is the Wright type function given in (3.8), and where the Sobolev type resolvent family $\{S^E_{2,1}(t)\}_{t\geq 0}$ is given in Proposition 7.

Similarly, we have:

Proposition 9. *Let $1 < \alpha < 2$. Let E be the multiplication operator by m, $X = L^2(\Omega)$ and $A = \Delta$ with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Then, the pair (A,E) is the generator of the (α,α)-Sobolev type resolvent family $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ given by*

$$
(5.12) \qquad S^E_{\alpha,\alpha}(t)x := \int_0^\infty \psi_{\frac{\alpha}{2},\frac{\alpha}{2}}(t,s)S^E_{2,1}(s)x\,ds, \quad t \geq 0, \ x \in X,
$$

where $\psi_{\frac{\alpha}{2},\frac{\alpha}{2}}$ is the Wright type function given in (3.10), and where the Sobolev type resolvent family $\{S^E_{2,1}(t)\}_{t\geq 0}$ is given in Proposition 7.

Finally, we have the following results for the existence of mild solutions in the case $1 < \alpha < 2$.

Theorem 10. *If $1 < \alpha < 2$, then the problem (5.8) for the Caputo fractional derivative has a unique mild solution given by*

$$
u(t) = S^E_{\alpha,1}(t)u_0 + (g_1 * S^E_{\alpha,1})(t)u_1 + \int_0^t (g_{\alpha-1} * S^E_{\alpha,1})(t-s)f(s)\,ds,
$$

where $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ is given in Proposition 8.

Proof. By Proposition 8, the pair (A,E) is the generator of the $(\alpha,1)$-Sobolev type resolvent family $\{S^E_{\alpha,1}(t)\}_{t\geq 0}$ given by (5.11) and then, the result holds by Theorem 3. $\square$

The proof of the next result follows from Theorem 4 and similarly to the proof of Theorem 10.

Theorem 11. *If* $1 < \alpha < 2$, *then the problem* (5.9) *for the Riemann-Liouville fractional derivative has a unique mild solution given by*

$$u(t) = S^E_{\alpha,\alpha}(t)u_1 + \int_0^t S^E_{\alpha,\alpha}(t-s)f(s)ds,$$

where $\{S^E_{\alpha,\alpha}(t)\}_{t\geq 0}$ *is given in Proposition 9.*

Acknowledgements

The author thanks to the anonymous referees for their carefully reading of the manuscript and for making suggestions which have improved the previous version of this paper.

References

[1] L. Abadias, P. J. Miana, A subordination principle on Wright functions and regularized resolvent families. *J. of Function Spaces* **2015** (2015) Article ID 158145, 9 pages.

[2] R. Agarwal, D. Baleanu, J. J. Nieto, D. Torres, Y. Zhou, A survey on fuzzy fractional differential and optimal control nonlocal evolution equations. *J. Comput. Appl. Math.* **339** (2018), 3–29.

[3] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems.* Monogr. Math., Vol. **96**, Birkhäuser, Basel (2011).

[4] E. Bazhlekova, *Fractional Evolution Equations in Banach Spaces.* Ph.D. Thesis, Eindhoven University of Technology (2001).

[5] E. Bazhlekova, Subordination in a class of generalized time-fractional diffusion-wave equations. *Fract. Calc. Appl. Anal.* **21**, No 4 (2018), 869–900; DOI: 10.1515/fca-2018-0048; https://www.degruyter.com/view/j/fca.2018.21.issue-4/fca-2018-0048/fca-2018-0048.xml.

[6] E. Bazhlekova, Subordination principle for space-time fractional evolution equations and some applications. *Integral Transforms Spec. Funct.* **30**, No 6 (2019), 431–452.

[7] E. Bazhlekova, I. Bazhlekov, Subordination approach to multi-term time-fractional diffusion-wave equations. *J. Comput. Appl. Math.* **339** (2018), 179–192.

[8] E. Bazhlekova, I. Bazhlekov, Subordination approach to space-time fractional diffusion. *Mathematics* **7** (2019), Art. 415.

[9] K. Balachandran, E. Anandhi, J. Dauer, Boundary controllability of Sobolev-type abstract nonlinear integrodifferential systems. *J. Math. Anal. Appl.* **277** (2003), 446–464.

[10] K. Balachandran, S. Kiruthika, J. Trujillo, On fractional impulsive equations of Sobolev type with nonlocal condition in Banach spaces. *Comput. Math. Appl.* **62** (2011), 1157–1165.

[11] G. Barenblat, J. Zheltor, I. Kochiva, Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks. *J. Appl. Math. Mech.* **24** (1960), 1286–1303.

[12] A. Benchaabane, R. Sakthivel, Sobolev-type fractional stochastic differential equations with nonLipschitz coefficients. *J. Comput. Appl. Math.* **312** (2017), 65–73.

[13] A. Debbouche, J. J. Nieto, Sobolev type fractional abstract evolution equations with nonlocal conditions and optimal multi-controls. *Appl. Math. and Comp.* **245** (2014), 74–85.

[14] A. Debbouche, J. J. Nieto, D. Torres, Optimal solutions to relaxation in multiple control problems of Sobolev type with nonlocal nonlinear fractional differential equations, *J. Optim. Theory Appl.* **174** (2017), 7–31.

[15] A. Debbouche, D. Torres, Sobolev type fractional dynamic equations and optimal multi-integral controls with fractional nonlocal conditions. *Fract. Calc. Appl. Anal.* **18**, No 1 (2015), 95–121; DOI: 10.1515/fca-2015-0007; https://www.degruyter.com/view/j/fca.2015.18.issue-1/fca-2015-0007/fca-2015-0007.xml

[16] A. Favaron, A. Favini, Maximal time regularity for degenerate evolution integro-differential equations, *J. of Evol. Equations* **10** (2010), 377–412.

[17] A. Favini, A. Lorenzi, Identification problems for singular integro-differential equations of parabolic type II. *Nonlinear Anal.* **56** (2004), 879–904.

[18] A. Favini, A. Yagi, *Degenerate Differential Equations in Banach Spaces.* Pure and Applied Math., **15**, Dekker, New York-Basel-Hong Kong (1999).

[19] M. Feckan, J. Wang, Y. Zhou, Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators. *J. Optim. Theory Appl.* **156** (2013), 79–95.

[20] V. Keyantuo, C. Lizama, M. Warma, Spectral criteria for solvability of boundary value problems and positivity of solutions of time-fractional differential equations, *Abstr. Appl. Anal.* **2013** (2013), Art. ID 614328, 11 pp.

[21] M. Li, C. Chen, F. Li, On fractional powers of generators of fractional resolvent families. *J. Funct. Anal.* **259** (2010), 2702–2726.

[22] F. Li, J. Liang, H. K. Xu, Existence of mild solutions for fractional integrodifferential equations of Sobolev type with nonlocal conditions. *J. Math. Anal. Appl.* **391** (2012), 510–525.

[23] K. Li, J. Peng, J. Jia, Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives. *J. Funct. Anal.* **263**, No 2 (2012), 476–510.

[24] J. Liang, T. Xiao, Abstract degenerate Cauchy problems in locally convex spaces, *J. Math. Anal. Appl.* **259** (2001), 398–412.

[25] J. Lightbourne III, S. Rankin III, A partial functional-differential equation of Sobolev type. *J. Math. Anal. Appl.* **93** (1983), 328–337.

[26] C. Lizama, Regularized solutions for abstract Volterra equations. *J. Math. Anal. Appl.* **243** (2000), 278–292.

[27] Y. Luchko, Subordination principles for the multi-dimensional space-time-fractional diffusion-wave equation (with English, Russian, Ukrainian summary). *Teor. Imovir. Mat. Stat.* No 98 (2018), 121–141; Reprinted in: *Theory Probab. Math. Statist.* **98** (2019), 127–147.

[28] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. Imperial College Press, (2010).

[29] G. Marinoschi, *Functional Approach to Nonlinear Models of Water Flow in Soils*. Mathematical Modeling, Theory and Applications, Vol. 21, Springer, Dordrecht (2006).

[30] Z. Mei, J. Peng, Y. Zhang, An operator theoretical approach to Riemann-Liouville fractional Cauchy problem. *Math. Nachr.* **288** (2015), 784–797.

[31] K. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York (1993).

[32] H. Pollard, The completely monotonic character of the Mittag-Leffler function $E_a(-x)$. *Bull. Amer. Math. Soc.* **54** (1948), 1115–1116.

[33] G. Sviridyuk, V. Fedorov, *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*. De Gruyter (2003).

[34] T. Xiao, J. Liang, *The Cauchy Problem for Higher-order Abstract Differential Equations*. Lecture Notes in Mathematics 1701, Springer (1998).

Instituto de Matemática y Física
Universidad de Talca
Casilla 747, Talca, CHILE

e-mail: rponce@inst-mat.utalca.cl