ON THE WELL-POSEDNESS OF DEGENERATE FRACTIONAL DIFFERENTIAL EQUATIONS IN VECTOR VALUED FUNCTION SPACES.

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ABSTRACT. We characterize completely the well-posedness on the vector-valued Hölder and Lebesgue spaces of the fractional degenerate differential equation $D^\alpha(Mu)(t) = Au(t) + f(t), t \in \mathbb{R}$ by using vector-valued multiplier results in the spaces $C^\gamma(\mathbb{R}; X)$ and $L^p(\mathbb{R}; X)$, where A and M are closed linear operators defined on the Banach space X, $0 < \gamma < 1$, $1 < p < \infty$, the fractional derivative is understood in the sense of Caputo and α is positive.

1. INTRODUCTION

Let A and M be two closed linear operators defined on a Banach space X with domains $D(A)$ and $D(M)$, respectively. In this paper, we study the maximal regularity of solutions for the following degenerate (also called *Sobolev*) type differential equation

$$D^\alpha(Mu(t)) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where the domains of A and M satisfy $D(A) \cap D(M) \neq \{0\}$ and the fractional derivative for $\alpha > 0$ is taken in the sense of Caputo, the function $f : \mathbb{R} \to X$ belongs to certain vector valued function space $S(\mathbb{R}, X)$.

A large number of partial differential equations arising in physics and applied sciences such as in the flow of fluid through fissured rocks, thermodynamics and shear in second order fluids or in the theory of control of dynamical systems can be expressed by the model in the form of (1.1). The most typical example is when $A = \Delta$ is the Laplacian and $M = m$ is the multiplication operator by a function $m(x)$, then the degenerate differential equation (in case $\alpha = 1$) describes the infiltration of water in unsaturated porous media, in which saturation might occur. See [20] for further details.

A detailed study of linear abstract Sobolev (or degenerate) type differential equations (1.1) (in case $\alpha \in \mathbb{N}$), has been described in the monographs by Favini and Yagi [23] and by Sviridyuk and Fedorov [33].

We notice that the well-posedness of degenerate differential equations has been recently studied. The first order problem, that is, when $\alpha = 1$ by S. Bu [13] in the Hölder continuous function spaces $C^\gamma(\mathbb{R}; X)$ ($0 < \gamma < 1$) (see also [30]). Whereas the well-posedness in the periodic Lebesgue spaces $L^p([0, 2\pi]; X)$ (where $1 < p < \infty$ and X is a UMD Banach space) can be found in [12, 25] and [26].

On the other hand, the well-posedness of the second order degenerate problem, that is, the case $\alpha = 2$, has been studied in the periodic L^p spaces $L^p([0, 2\pi]; X)$, the periodic Besov spaces $B^s_{p,q}([0, 2\pi]; X)$, the Triebel-Lizorkin spaces $F^s_{p,q}([0, 2\pi]; X)$ and the Hölder continuous function space $C^\gamma(\mathbb{R}; X)$, by S. Bu and G. Cai in [11, 14] and [15]. See also [32] for the case $L^p(\mathbb{R}; X)$ and [12] for the delay equation. Finally, and very recently, S. Bu and G. Cai characterize completely the well-posedness of the third order equation [16] in the spaces $L^p([0, 2\pi]; X)$, $B^s_{p,q}([0, 2\pi]; X)$, and $F^s_{p,q}([0, 2\pi]; X)$. We notice that the main tool in the study of the well-posedness of the degenerate differential equations are operator-valued multiplier theorems established in [2, 4, 5] and [17].

We notice that, the problem of characterize the well-posedness (or maximal regularity) of abstract fractional differential equations has been studied intensively in the last years for $0 < \alpha \leq 2$, in case $M = I$, (the identity operator) in [8, 9, 10, 24, 28] and [29] in the spaces $L^p([0, 2\pi]; X)$, $L^p(\mathbb{R}; X)$ and $C^\gamma(\mathbb{R}; X)$. On the other hand, we observe that even though the change of variable $v(t) = Mu(t)$ reduces

2000 *Mathematics Subject Classification.* 34G10, 34K30, 35K65.

Key words and phrases. Degenerate fractional equations; Maximal regularity; Operator-valued Fourier multipliers; Sobolev type equations.

the problem (1.1) to the multivalued fractional differential equation

$$D^{\alpha}v(t) \in Lv(t) + f(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $L = AM^{-1}$ and $D(L) = M(D(A))$ and therefore the equation (1.1) can be written formally as equation (1.2), that is, the the fractional differential equation studied in [8, 9, 10, 24, 28] and [29], we can not apply the results in the above mentioned papers to problem (1.2) to obtain the well-posedness of equation (1.1) because this results are valid only in the single-valued case.

In this paper, we study the well-posedness (or the maximal regularity property) of the equation (1.1) in the Hölder spaces $C^{\gamma}(\mathbb{R}; X)$ $(0 < \gamma < 1)$, and the Lebesgue-Bochner $L^p(\mathbb{R}, X)$ without assuming that M has bounded (or compact) inverse.

The paper is organized as follows. Section 2 collects the preliminaries and some results about Caputo fractional calculus. Section 3 is devoted to the C^{γ}-well-posedness of equation (1.1), that is, the well-posedness in the Hölder continuous function space $C^{\gamma}(\mathbb{R}; X)$ where $0 < \gamma < 1$ and X is a Banach space. We remark that in the results obtained in this section, there is not conditions on commutativity of A with M, or in the existence of bounded inverse of A or M. In Section 4 we study the well-posedness (or maximal regularity) of equation (1.1) in the Lebesgue space $L^p(\mathbb{R}; X)$ where $1 < p < \infty$ and X is a UMD space. Finally, in Section 5, we give some examples.

2. Preliminaries

Let X and Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ be the space of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X, we denote domain by $D(A)$ and its resolvent set by $\rho(A)$. By $[D(A)]$ we denote the domain of A equipped with the graph norm.

The *M-modified resolvent set of A*, $\rho_M(A)$, is defined by

$$\rho_M(A) := \{\lambda \in \mathbb{C} : (\lambda M - A) : D(A) \cap D(M) \to X$$
$$\text{is invertible and } (\lambda M - A)^{-1} \in \mathcal{B}(X, [D(A) \cap D(M)])\}.$$

We denote by $\mathcal{F}f$ the Fourier transform of f, that is

$$(\mathcal{F}f)(s) := \tilde{f}(s) := \int_{\mathbb{R}} e^{-ist} f(t)dt,$$

for $s \in \mathbb{R}$ and $f \in L^1(\mathbb{R}; X)$.

The Laplace transform of a function $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$ is denoted by

$$\hat{f}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t)dt, \quad \text{Re}\lambda > \omega,$$

whenever the integral is absolutely convergent for $\text{Re}\lambda > \omega$. The relation between the Laplace transform of $f \in L^1(\mathbb{R}; X)$, $f(t) = 0$ for $t < 0$, and its Fourier transform is

$$\mathcal{F}(f)(s) = \hat{f}(is), \quad s \in \mathbb{R}.$$

For $f \in L^1_{\text{loc}}(\mathbb{R}; X)$ of subexponential growth, that is

$$\int_{-\infty}^{\infty} e^{-\epsilon|t|} \|f(t)\| dt < \infty, \qquad \text{for each } \epsilon > 0,$$

we denote by $\hat{f}(\lambda)$ for the *Carleman transform of f* :

$$\hat{f}(\lambda) = \begin{cases} \int_0^{\infty} e^{-\lambda t} f(t)dt, & \text{Re}\lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t} f(t)dt, & \text{Re}\lambda < 0. \end{cases}$$

Observe that we use the same symbol for the Carleman and Laplace transform but, this will not lead to confusion.

Given $\alpha > 0$, the *Liouville fractional integrals of order α*, $D_-^{-\alpha}f$ and $D_+^{-\alpha}f$ are defined, respectively, by

$$D_-^{-\alpha}f(t) := \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad t \in \mathbb{R}, \tag{2.1}$$

and

$$D_+^{-\alpha} f(t) := \int_t^\infty \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad t \in \mathbb{R}. \tag{2.2}$$

A sufficient condition for that the fractional integrals (2.1) and (2.2) exist is that $f(t) = O(|t|^{-\alpha-\epsilon})$ for $\epsilon > 0$ and $t \to \infty$. Integrable functions satisfying this property are sometimes referred to as functions of Liouville class, see [27].

The *Caputo left and right-sided fractional derivatives,* corresponding to those in (2.1) and (2.2) are defined, respectively, by

$$D_-^\alpha f(t) := D_-^{-(n-\alpha)} \frac{d^n}{dt^n} f(t) = \int_{-\infty}^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s)ds \tag{2.3}$$

and

$$D_+^\alpha f(t) := (-1)^n D_+^{-(n-\alpha)} \frac{d^n}{dt^n} f(t) = (-1)^n \int_t^\infty \frac{(s-t)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s)ds, \tag{2.4}$$

where $t \in \mathbb{R}$, $f \in C^n(\mathbb{R}; X)$ and $n = \lceil \alpha \rceil$. Here $\lceil \alpha \rceil$ denotes the the smallest integer greater than or equal to α.

It is known that $D_\pm^{\alpha+\beta} = D_\pm^\alpha (D_\pm^\beta)$ for any $\alpha, \beta \in \mathbb{R}$, where $D_\pm^0 = \mathrm{Id}$ denotes the identity operator and $(-1)^n D_+^n = D_-^n = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$. See [27].

In what follows, we refer to the Caputo left-sided fractional derivative, $D_-^\alpha f$, as the *Caputo fractional derivative of order* $\alpha > 0$ *of* f and we write $D^\alpha f := D_-^\alpha f$. For example, for the function $e^{\lambda t}$ we have

$$D_-^{-\alpha} e^{\lambda t} = \lambda^{-\alpha} e^{\lambda t} \quad \text{and} \quad D^\alpha e^{\lambda t} = \lambda^\alpha e^{\lambda t}, \quad \mathrm{Re}\lambda \geq 0.$$

The Caputo left and right-sided fractional derivatives are adjoint in the sense of the following lemma.

Lemma 2.1. [29] *If $D^\alpha f$ and $D_+^{-\alpha} g$ exist, then*

$$\int_{\mathbb{R}} f(t)g(t)dt = \int_{\mathbb{R}} D^\alpha f(t) D_+^{-\alpha} g(t)dt.$$

3. C^γ-Well-posedness

Let $0 < \gamma < 1$. We denote by $C^\gamma(\mathbb{R}; X)$ the space of all X-valued functions f on $\mathbb{R}$, such that

$$\|f\|_\gamma := \sup_{t \neq s} \frac{\|f(t) - f(s)\|}{|t-s|^\alpha} < \infty.$$

If we define $\|f\|_{C^\gamma} := \|f\|_\gamma + \|f(0)\|$, then $C^\gamma(\mathbb{R}; X)$ is a Banach space under the norm $\|\cdot\|_{C^\gamma}$.

The kernel of the seminorm $\|\cdot\|_\gamma$ on $C^\gamma(\mathbb{R}; X)$ is the space of all constant functions and the corresponding quotient space $\dot{C}^\gamma(\mathbb{R}; X)$ is a Banach space in the induced norm. We identify a function $f \in C^\gamma(\mathbb{R}; X)$ with its equivalence class

$$\dot{f} := \{g \in C^\gamma(\mathbb{R}; X) : f - g \equiv \text{ constant}\}.$$

In this way, $\dot{C}^\gamma(\mathbb{R}; X)$ may be identified with the space of all $f \in C^\gamma(\mathbb{R}; X)$ such that $f(0) = 0$. See [2, Section 5].

For $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, $C^n(\mathbb{R}; X)$ denotes the set of X-valued functions which are n-times differentiable on $\mathbb{R}$.

For $\alpha > 0$, let $C^{\alpha,\gamma}(\mathbb{R}, X)$ be the Banach space of all $u \in C^n(\mathbb{R}, X)$, $n = \lceil \alpha \rceil$, such that $D^\alpha u$ exists and belongs to $C^\gamma(\mathbb{R}, X)$ equipped with the norm

$$\|u\|_{C^{\alpha,\gamma}} = \|D^\alpha u\|_{C^\gamma} + \sum_{j=1}^n \|D^{\alpha-j} u(0)\|.$$

Let $\Omega \subset \mathbb{R}$ be an open set. By $C_c^\infty(\Omega)$ we denote the space of all $C^\infty-$functions in Ω having compact support in Ω.

Definition 3.1. *Let $M : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X,Y)$ be continuous. We say that $\mathcal{M}$ is a $\dot{C}^{\gamma}-$*multiplier *if there exists a mapping $L : \dot{C}^{\gamma}(\mathbb{R};X) \to \dot{C}^{\gamma}(\mathbb{R};Y)$ such that*

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot \mathcal{M}))(s)f(s)ds \tag{3.1}$$

for all $f \in C^{\gamma}(\mathbb{R};X)$ and all $\phi \in C_c^{\infty}(\mathbb{R} \setminus \{0\})$.

Here $(\mathcal{F}(\phi \cdot \mathcal{M}))(s) = \int_{\mathbb{R}} e^{-ist}\phi(t)\mathcal{M}(t)dt \in \mathcal{B}(X,Y)$. Observe that the right-hand side of (3.1) does not depend on the representative of $\dot{f}$ since

$$\int_{\mathbb{R}} (\mathcal{F}(\phi\mathcal{M})(s))(s)ds = 2\pi(\phi\mathcal{M})(0) = 0.$$

Therefore, if L exists, then it is well defined. Moreover, left-hand side of (3.1) determines the function $Lf \in C^{\gamma}(\mathbb{R};X)$ uniquely up to some constant (by [2, Lemma 5.1]). Moreover, if (3.1) holds, then $L : \dot{C}^{\gamma}(\mathbb{R};X) \to \dot{C}^{\gamma}(\mathbb{R};Y)$ is linear and continuous (see [2, Definition 5.2]) and if $f \in C^{\gamma}(\mathbb{R};X)$ is bounded, then Lf is bounded as well (see [2, Remark 6.3]).

The following multiplier theorem is due to Arendt, Batty and Bu [2, Theorem 5.3].

Theorem 3.2. *Let $\mathcal{M} \in C^2(\mathbb{R} \setminus \{0\}, \mathcal{B}(X,Y))$ be such that*

$$\sup_{t\neq 0} \|\mathcal{M}(t)\| + \sup_{t\neq 0} \|t\mathcal{M}'(t)\| + \sup_{t\neq 0} \|t^2\mathcal{M}''(t)\| < \infty. \tag{3.2}$$

Then, $\mathcal{M}$ is a $\dot{C}^{\gamma}-$multiplier.

Example 3.3. *Let X be an Banach space and $0 < \gamma < 1$. Define $N(t) = I$ for $t \geq 0$ and $N(t) = 0$ for $t < 0$. It follows from Theorem 4.17 that N is a $\dot{C}^{\gamma}$-multiplier. The associated operator on $\dot{C}^{\gamma}(\mathbb{R};X)$ is called the* Riesz projection.

Example 3.4. *Let X be an Banach space and $0 < \gamma < 1$. Define $N(t) = (-i\,\mathrm{sign}t)I$ for $t \in \mathbb{R}$. Then N is a $\dot{C}^{\gamma}$-multiplier by Theorem 4.17. The associated operator on $\dot{C}^{\gamma}(\mathbb{R};X)$ is called the* Hilbert transform.

Remark 3.5.

Recall that a Banach space X has the *Fourier type* p, with $1 \leq p \leq 2$, if the Fourier transform defines a bounded linear operator from $L^p(\mathbb{R};X)$ to $L^q(\mathbb{R};X)$, where $1/p + 1/q = 1$. As examples, $L^p(\Omega)$, with $1 \leq p \leq 2$, has Fourier type p; the Banach space X has the Fourier type 2 if and only if X is isomorphic to a Hilbert space; X has Fourier type p if and only if X^* has Fourier type p. Every Banach space has Fourier type 1. A Banach space X is say to be $B-$convex if it has Fourier type p, for some $p > 1$. Every uniformly convex space is $B-$convex.

If X is $B-$convex, in particular if X is a UMD space, then $\mathcal{M} \in C^1(\mathbb{R}\setminus\{0\}, \mathcal{B}(X,Y))$ is a $\dot{C}^{\gamma}-$*multiplier* if the condition (3.2) is replaced by the weaker condition

$$\sup_{t\neq 0} \|\mathcal{M}(t)\| + \sup_{t\neq 0} \|t\mathcal{M}'(t)\| < \infty, \tag{3.3}$$

see [2, Remark 5.5].

We conclude this section with the following Lemmas.

Lemma 3.6. [2] *Let $f \in C^{\gamma}(\mathbb{R};X)$. Then f is constant if and only if $\int_{\mathbb{R}} f(s)(\mathcal{F}\varphi)(s)ds = 0$ for all $\varphi \in C_c^{\infty}(\mathbb{R} \setminus \{0\})$.*

Lemma 3.7. [29] *Let $0 < \gamma < 1$, $u, v \in C^{\gamma}(\mathbb{R};X)$ and $\alpha > 0$. Then, the following assertions are equivalent,*

(i) $u \in C^{\alpha,\gamma}(\mathbb{R};X)$ and $D^{\alpha}u - v$ is constant;
(ii) $\int_{\mathbb{R}} v(s)\mathcal{F}(\phi)(s)ds = \int_{\mathbb{R}} u(s)\mathcal{F}(\mathrm{id}^{\alpha} \cdot \phi)(s)ds$, for all $\phi \in C_c^{\infty}(\mathbb{R} \setminus \{0\})$.

As in [2] we define the map $\mathrm{id} : \mathbb{R} \to \mathbb{C}$ by $\mathrm{id}(s) = is$. The function id^{α} is defined by $\mathrm{id}^{\alpha}(s) = (is)^{\alpha}$, where $(is)^{\alpha} = |s|^{\alpha}e^{\frac{\pi\alpha i}{2}\mathrm{sgn}(s)}$ (here $\mathrm{sgn}(s)$ denotes the sign of s).

Now, we consider the degenerate fractional differential equation

$$D^{\alpha}(Mu)(t) = Au(t) + f(t), \qquad t \in \mathbb{R}, \tag{3.4}$$

where $A : D(A) \subseteq X \to X$ and $M : D(M) \subseteq X \to X$ are closed linear operators defined on X, with $D(A) \cap D(M) \neq \{0\}$, and $f \in C^{\gamma}(\mathbb{R};X)$, $0 < \gamma < 1$.

Definition 3.8. *We say that the equation (3.4) is* C^γ-well posed *if, for each* $f \in C^\gamma(\mathbb{R};X)$*, there exists a unique function* $u \in C^\gamma(\mathbb{R};[D(A)\cap D(M)])$ *such that* $Mu \in C^{\alpha,\gamma}(\mathbb{R};X)$ *and the equation (3.4) holds for all* $t \in \mathbb{R}$.

We define the set

$$H^{\alpha,\gamma}(\mathbb{R};[D(M)]) \;=\; \{u \in C(\mathbb{R};[D(M)]) : D^\alpha u \text{ exists and } \exists v \in C^\gamma(\mathbb{R};X) \text{ such that } v = D^\alpha(Mu)\}.$$

Remark 3.9.

Observe that if (3.4) is C^γ-well posed, it follows from the closed graph theorem that the map $L : C^\gamma(\mathbb{R};X) \to C^\gamma(\mathbb{R};[D(A)\cap D(M)]) \cap H^{\alpha,\gamma}(\mathbb{R};[D(M)])$ which associates to the function f the unique solution u of (3.4) is linear and continuous. Indeed, since A, M are closed operators, we have that the space $H := C^\gamma(\mathbb{R};[D(A)\cap D(M)]) \cap H^{\alpha,\gamma}(\mathbb{R};[D(M)])$ endowed with the norm

$$\|u\|_H := \|D^\alpha(Mu)\|_{C^\gamma} + \|Au\|_{C^\gamma} + \|u\|_{C^\gamma}$$

is a Banach space.

We begin with the following result.

Proposition 3.10. *Let* $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ *closed linear operators defined on a Banach space* X *satisfying* $D(A)\cap D(M) \neq \{0\}$*. Suppose that the problem (3.4) is* C^γ*-well posed. Then,*

(*i*) $(is)^\alpha \in \rho_M(A)$, *for all* $s \in \mathbb{R}$, *and*

(*ii*) $\sup_{s\in\mathbb{R}} ||(is)^\alpha M((is)^\alpha M - A)^{-1}|| < \infty$.

Proof. Let $s \in \mathbb{R}$ and suppose that

$$((is)^\alpha M - A)x = 0 \tag{3.5}$$

where $x \in D(A)\cap D(M)$. Let $u(t) = e^{ist}x$. Then, u is a solution to (3.4) with $f \equiv 0$. In fact, since $D^\alpha(Mu)(t) = (is)^\alpha e^{ist}Mx$ (see [27, p. 248]) we have by (3.5)

$$Au(t) = e^{ist}Ax = e^{ist}(is)^\alpha Mx = D^\alpha(Mu)(t).$$

Hence, by uniqueness it follows that $u \equiv 0$, that is, $x = 0$. We conclude that $((is)^\alpha M - A)$ is injective. Now we prove the surjectivity, let $y \in X$. Let $L : C^\gamma(\mathbb{R};X) \to C^\gamma(\mathbb{R};[D(A)\cap D(M)]) \cap H^{\alpha,\gamma}(\mathbb{R};[D(M)])$ be the bounded linear operator which takes each $f \in C^\gamma(\mathbb{R};X)$ to the unique solution u of equation (3.4). Let $s \in \mathbb{R}$, $f(t) = e^{ist}y$ and $u = Lf$. Take $s_0 \in \mathbb{R}$ fixed. We have that $v_1(t) := u(t+s_0)$ and $v_2(t) := e^{is_0s}u(t)$ are both solutions of (3.4) with $g(t) = e^{is_0s}f(t)$. Hence, $v_1 = v_2$, that is, $u(t+s_0) = e^{is_0s}u(t)$ for all $s_0, t \in \mathbb{R}$. Let $x = u(0) \in D(A)\cap D(M)$. Since $u(t) = e^{ist}x$ solves the equation (3.4) for all $t \in \mathbb{R}$, we have in particular for $t = 0$, that,

$$((is)^\alpha M - A)x = (is)^\alpha Mu(0) - Au(0) = D^\alpha(Mu)(0) - Au(0) = f(0) = y. \tag{3.6}$$

Therefore $((is)^\alpha M - A)$ is surjective for all $s \in \mathbb{R}$. By (3.6), we have $x = ((is)^\alpha M - A)^{-1}$ and thus

$$\|((is)^\alpha M - A)^{-1}y\| = \|x\| = \|u(0)\| = \|Lf(0)\| \le \|L\|\|f(0)\| = \|L\|\|y\|,$$

for all $y \in X$. We conclude that $((is)^\alpha M - A)^{-1}$ is a bounded operator and therefore $(is)^\alpha \in \rho_M(A)$ for all $s \in \mathbb{R}\setminus\{0\}$.

On the other hand, by (3.6) we have $u(t) = e^{ist}((is)^\alpha M - A)^{-1}y$. Denote by $e_s \otimes x$ the function $t \to (e_s \otimes x)(t) := e^{ist}x$. Since $||e_s \otimes x||_\gamma = C_\gamma|s|^\gamma||x||$, where $C_\gamma = 2\sup_{t>0} t^{-\gamma}\sin(t/2)$ (see [2, Section 3]), we have

$$\begin{aligned} C_\gamma|s|^\gamma\|(is)^\alpha M((is)^\alpha M - A)^{-1}y\| &= \|(is)^\alpha e_s \otimes M((is)^\alpha M - A)^{-1}y\|_\gamma = \|D^\alpha(Mu)\|_\gamma \\ &\le \|Mu\|_{C^{\alpha,\gamma}} \le \|L\|\,\|f\|_{C^\gamma} \\ &= \|L\|(C_\gamma|s|^\gamma + 1)\|y\|. \end{aligned}$$

Hence,

$$\|(is)^\alpha M((is)^\alpha M - A)^{-1}y\| \le \|L\|\left(1 + C_\alpha^{-1}|s|^{-\gamma}\right)\|y\|.$$

Thus $(is)^\alpha M((is)^\alpha M - A)^{-1}$ is a bounded operator for every $s \in \mathbb{R}\setminus\{0\}$. It remains to show that $0 \in \rho_M(A)$. If $s = 0$ then f is the constant function y, and the corresponding solution to (3.4) is the constant function $-A^{-1}y \in D(A)\cap D(M)$. Therefore

$$\|A^{-1}y\| = \|u(0)\| \le \|u\|_{H^{\alpha,\gamma}(\mathbb{R};[D(M)])} \le \|L\|\|f\|_{C^\gamma(\mathbb{R};X)} = \|L\|\|f\|,$$

and thus, $0 \in \rho_M(A)$. We conclude that $(is)^\alpha \in \rho_M(A)$ for all $s \in \mathbb{R}$, and $(is)^\alpha M((is)^\alpha M - A)^{-1}$ is a bounded operator for every $s \in \mathbb{R}$. The proof is complete.

■

In the following Theorem we prove a converse of Proposition 3.10.

Theorem 3.11. *Let $0 < \gamma < 1$ and $1 \leq \alpha \leq 2$. Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ closed linear operators defined on a Banach space X satisfying $D(A) \cap D(M) \neq \{0\}$. Then, the following assertions are equivalent*

(i) The equation (3.4) is C^γ-well posed;

(ii) $(is)^\alpha \in \rho_M(A)$ for all $s \in \mathbb{R}$ and $\sup\limits_{s\in\mathbb{R}} \|(is)^\alpha M((is)^\alpha M - A)^{-1}\| < \infty$.

Proof. $(ii) \Rightarrow (i)$. For $s \in \mathbb{R}$, we define the operator $N(s) := ((is)^\alpha M - A)^{-1}$. Observe that by hypothesis $N \in C^2(\mathbb{R}; \mathcal{B}(X, [D(A) \cap D(M)]))$. We claim that N is a $\dot{C}^\alpha$-multiplier. In fact, since $0 \in \rho_M(A)$, we have that A^{-1} is bounded (seen as an operator from X to $[D(A) \cap D(M)]$). The identity $(is)^\alpha MN(s) - I = AN(s)$ shows that $A^{-1}((is)^\alpha MN(s) - I) = N(s)$ and therefore, by hypothesis we obtain $\sup_{s\in\mathbb{R}} \|N(s)\| < \infty$. On the other hand,

$$sN'(s) = -\alpha(is)^\alpha MN(s)N(s),$$

$$s^2N''(s) = -\alpha(\alpha-1)(is)^\alpha MN(s)N(s) - 2\alpha(is)^\alpha MN(s)sN'(s).$$

The hypothesis we have $\sup_{s\in\mathbb{R}} \|(is)^\alpha MN(s)\| < \infty$ and the above identities, show that

$$\sup_{s\in\mathbb{R}} ||sN'(s)|| < \infty \quad \text{and} \quad \sup_{s\in\mathbb{R}} ||s^2N''(s)|| < \infty.$$

We conclude from Theorem 4.17 that N is a $\dot{C}^\gamma$-multiplier.

Define $S(s) := (\mathrm{id}^\alpha \cdot MN)(s)$, where $\mathrm{id}^\alpha(s) = (is)^\alpha$ and $s \in \mathbb{R}$. Observe that by hypothesis $S \in C^2(\mathbb{R}; \mathcal{B}(X))$ and $\sup_{s\in\mathbb{R}} \|S(s)\| < \infty$. Moreover,

$$\begin{aligned} sS'(s) &= \alpha(is)^\alpha MN(s) - \alpha(is)^\alpha MN(s)(is)^\alpha MN(s) \\ &= \alpha S(s) - \alpha S(s)S(s). \end{aligned}$$

We conclude that $\sup_{s\in\mathbb{R}} \|sS'(s)\| < \infty$. Since $(sS'(s))' = S''(s) + sS'(s)$ and $sS'(s) = \alpha S(s) - \alpha S^2(s)$ we obtain that $s^2S''(s) = \alpha sS'(s) - 2\alpha S(s)sS'(s) - sS'(s)$ and therefore $\sup_{s\in\mathbb{R}} \|s^2S''(s)\| < \infty$. Hence, from hypothesis $\sup_{s\in\mathbb{R}} ||S(s)|| < \infty$, $\sup_{s\in\mathbb{R}} ||sS'(s)|| < \infty$ and $\sup_{s\in\mathbb{R}} ||s^2S''(s)|| < \infty$. We conclude that S is a $\dot{C}^\gamma$-multiplier by Theorem 4.17.

Let $f \in C^\gamma(\mathbb{R}; X)$. Since N and S are $\dot{C}^\gamma$-multipliers, there exist $\overline{u} \in C^\gamma(\mathbb{R}; [D(A) \cap D(M)])$, and $v \in C^\gamma(\mathbb{R}; X)$ such that

$$\int_{\mathbb{R}} \overline{u}(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi \cdot N)(s)f(s)ds, \tag{3.7}$$

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot S)(s)f(s)ds, \tag{3.8}$$

for all $\phi, \varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Take $\phi = \mathrm{id}^\alpha \cdot \varphi$ in (3.7). We obtain using (3.8)

$$\int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(\mathrm{id}^\alpha \cdot \varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\mathrm{id}^\alpha \cdot \varphi \cdot N)(s)f(s)ds, \tag{3.9}$$

for all $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

On the other hand, observe that $\overline{u}(t) \in D(A) \cap D(M)$ and $\mathcal{F}(\phi \cdot N)(s)x \in D(A) \cap D(M)$ for all $x \in X$, $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Using the fact that M is closed with $D(A) \cap D(M) \neq \{0\}$, we have from (3.7), (3.8) and (3.9) that

$$\begin{aligned}
&\int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id}^\alpha\cdot\varphi)(s)ds - A\int_{\mathbb{R}}\overline{u}(s)\mathcal{F}(\varphi)(s)ds = \\
&\quad = \int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id}^\alpha\cdot\varphi)(s)ds - A\int_{\mathbb{R}}\mathcal{F}(\varphi\cdot N)(s)f(s)ds \\
&\quad = \int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id}^\alpha\cdot\varphi)(s)ds - \int_{\mathbb{R}}\mathcal{F}(\varphi\cdot AN)(s)f(s)ds \\
&\quad = \int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id}^\alpha\cdot\varphi)(s)ds - \int_{\mathbb{R}}\mathcal{F}(\varphi\cdot[\mathrm{id}^\alpha MN - I])(s)f(s)ds \\
&\quad = \int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id}^\alpha\cdot\varphi)(s)ds - \int_{\mathbb{R}}\mathcal{F}(\varphi\cdot\mathrm{id}^\alpha MN)(s)f(s)ds + \int_{\mathbb{R}}\mathcal{F}(\varphi\cdot I)(s)f(s)ds \\
&\quad = \int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id}^\alpha\cdot\varphi)(s)ds - M\int_{\mathbb{R}}\overline{u}(s)\mathcal{F}(\varphi\cdot\mathrm{id}^\alpha)(s)ds + \int_{\mathbb{R}}\mathcal{F}(\varphi\cdot I)(s)f(s)ds \\
&\quad = \int_{\mathbb{R}}\mathcal{F}(\varphi\cdot I)(s)f(s)ds.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id}^\alpha\cdot\varphi)(s)ds = A\int_{\mathbb{R}}\overline{u}(s)\mathcal{F}(\varphi)(s)ds + \int_{\mathbb{R}}\mathcal{F}(\varphi\cdot I)(s)f(s)ds, \tag{3.10}$$

for all $\varphi \in C_c^\infty(\mathbb{R}\setminus\{0\})$.

Moreover, from (3.8) and (3.9) we have

$$\int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id}^\alpha\cdot\varphi)(s)ds = \int_{\mathbb{R}}\mathcal{F}(\varphi\cdot S)(s)f(s)ds = \int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds. \tag{3.11}$$

Since $\overline{u}\in C^\gamma(\mathbb{R};[D(A)\cap D(M)])$ and $D(A)\cap D(M)\neq\{0\}$, we have that $M\overline{u}\in C^\gamma(\mathbb{R};X)$. It follows from (3.11) and Lemma 3.7 that $D^\alpha(M\overline{u}) = v + y_1$ where $y_1\in X$. From (3.10) and (3.11) we have

$$\int_{\mathbb{R}} v(s)\mathcal{F}(\varphi)(s)ds = A\int_{\mathbb{R}}\overline{u}(s)\mathcal{F}(\varphi)(s)ds + \int_{\mathbb{R}}\mathcal{F}(\varphi\cdot I)(s)f(s)ds.$$

From Lemma 3.6 we obtain $v = A\overline{u} + f + y_2$ where $y_2\in X$. Therefore $D^\alpha(M\overline{u}) = A\overline{u} + f + y_3$ with $y_3 = y_1 + y_2$. Let $u(t) = \overline{u}(t) + x$ where $x = A^{-1}y_3$. Note that x is well defined since $i\mathbb{R}\subset\rho_M(A)$. Since $\overline{u}\in C^\gamma(\mathbb{R};[D(A)\cap D(M)])\cap H^{\alpha,\gamma}(\mathbb{R};[D(M)])$ we have $u\in C^\gamma(\mathbb{R};[D(A)\cap D(M)])\cap H^{\alpha,\gamma}(\mathbb{R};[D(M)])$. Since the fractional derivative (in the sense of Caputo) is zero, an easy computation shows that u satisfies the equation (3.4). The uniqueness follows similarly to [29, Theorem 3.7]

$(i)\Rightarrow(ii)$. Follows from Proposition 3.10. ∎

By Remark 3.5, we have the following result.

Theorem 3.12. *Let $0<\gamma<1$ and $0<\alpha\leq 1$. Let $A: D(A)\subseteq X\to X$, $M: D(M)\subseteq X\to X$ closed linear operators defined on a B-convex space X satisfying $D(A)\cap D(M)\neq\{0\}$. Then, the following assertions are equivalent*

(i) The equation (3.4) is C^γ-well posed;

(ii) $(is)^\alpha\in\rho_M(A)$ for all $s\in\mathbb{R}$ and $\sup_{s\in\mathbb{R}}\|(is)^\alpha M((is)^\alpha M - A)^{-1}\| < \infty$.

Corollary 3.13. *In the context of Theorem 3.11, if condition (ii) is fulfilled, we have $D^\alpha(Mu), Au\in C^\gamma(\mathbb{R};X)$. Moreover, there exists a constant $C>0$ independent of $f\in C^\gamma(\mathbb{R};X)$ such that*

$$||D^\alpha(Mu)||_{C^\gamma(\mathbb{R};X)} + ||Au||_{C^\gamma(\mathbb{R};X)} \leq C||f||_{C^\gamma(\mathbb{R};X)}. \tag{3.12}$$

Remark 3.14.

The inequality (3.12) is a consequence of the Closed Graph Theorem and known as the *maximal regularity property* for equation (3.4). We deduce that the operator S defined by:

$$(Su)(t) = D^\alpha(Mu)(t) - Au(t)$$

with domain

$$D(S) = H^{\alpha,\gamma}(\mathbb{R};[D(M)])\cap C^\gamma(\mathbb{R};[D(A)\cap D(M)]),$$

is an isomorphism onto. In fact, by Remark 3.9 we have that the space $H := C^\gamma(\mathbb{R};[D(A)\cap D(M)])\cap H^{\alpha,\gamma}(\mathbb{R};[D(M)])$ becomes a Banach space under the norm

$$||u||_H := ||u||_{C^\gamma(\mathbb{R};X)} + ||D^\alpha(Mu)||_{C^\gamma(\mathbb{R};X)} + ||Au||_{C^\gamma(\mathbb{R};X)}.$$

We remark that such isomorphisms are crucial for the handling of nonlinear evolution equations (see [1]). Indeed, assume that X is a Banach space and A, M satisfy the condition (ii) in Theorem 3.11. Consider the semilinear problem

$$D^\alpha(Mu)(t) = Au(t) + f(t,u(t)), \quad t\in\mathbb{R}. \tag{3.13}$$

Define the Nemytskii's superposition operator $N : H \to C^\gamma(\mathbb{R};X)$ given by $N(v)(t) = f(t, v(t))$ and the bounded linear operator

$$T := S^{-1} : C^\gamma(\mathbb{R};X) \to H$$

by $T(g) = u$ where u is the unique solution to linear problem

$$D^\alpha(Mu)(t) = Au(t) + g(t), \quad t \geq 0.$$

Then, to solve (3.13) we need to show that the operator $R : H \to H$ defined by $R = TN$ has a fixed point. For more details, we refer to H. Amann [1], H. Brill [7] and A. Rutkas and L. Vlasenko [31].

4. L^p-Well-posedness

Let $1 \leq p < \infty$. We denote by $L^p(\mathbb{R};X)$ the Banach space of all functions $f : \mathbb{R} \to X$, such that

$$\|f\|_p := \left(\int_{\mathbb{R}} \|f(t)\|^p dt\right)^{1/p} < \infty.$$

For $\alpha > 0$, we define $W^{\alpha,p}(\mathbb{R};X)$ as the Banach space consisting of all $u \in L^p(\mathbb{R};X)$, for which there exists $u', u'', ..., u^n \in L^p(\mathbb{R};X)$, $n = \lceil\alpha\rceil$, such that

$$\int_{\mathbb{R}} u(t)D^\alpha\phi(t)dt = \int_{\mathbb{R}} D^\alpha u(t)\phi(t)dt$$

for all $\phi \in \mathcal{D}(\mathbb{R})$.

Thus, if $u \in L^p(\mathbb{R};[D(A)\cap D(M)])$ is a *weak solution* of equation (1.1), i.e.

$$\int_{\mathbb{R}} Mu(t)D^\alpha\phi(t)dt = \int_{\mathbb{R}} (Au(t) + f(t))dt$$

for all $\phi \in \mathcal{D}(\mathbb{R})$, then $Mu \in W^{\alpha,p}(\mathbb{R};X)$ and $D^\alpha(Mu) = Au + f$.

We denote by $\mathcal{D}(\mathbb{R};X)$ the space of X-valued $C^\infty-$functions with compact support on $\mathbb{R}$. $\mathcal{S}'(\mathbb{R};X) = \mathcal{B}(\mathcal{S}(\mathbb{R});X)$ is the space of all tempered distributions. Then the Fourier transform $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R};X)$ is defined by

$$\langle\mathcal{F}u,\phi\rangle = \langle u,\hat{\phi}\rangle,$$

where $u \in \mathcal{S}'(\mathbb{R},X)$ and $\phi \in \mathcal{S}(\mathbb{R})$. If we identify $\mathcal{S}(\mathbb{R};X)$ with a subspace of $\mathcal{S}'(\mathbb{R};X)$ by letting

$$\langle u,\phi\rangle = \int_{\mathbb{R}} u(t)\phi(t)dt, \quad \phi \in \mathcal{S}(\mathbb{R}),$$

for all $u \in \mathcal{S}(\mathbb{R};X)$, then $\hat{u} = \mathcal{F}u$, i.e.,

$$\int_{\mathbb{R}} u(t)\hat{\phi}(t)dt = \int_{\mathbb{R}} \hat{u}(s)\phi(s)ds,$$

for all $u \in \mathcal{S}(\mathbb{R};X), \phi \in \mathcal{S}(\mathbb{R})$. Thus $\mathcal{F} : \mathcal{S}'(\mathbb{R};X) \to \mathcal{S}'(\mathbb{R};X)$ is an isomorphism extending the isomorphism $u \mapsto \hat{u}$ on $\mathcal{S}(\mathbb{R};X)$. See [1] for more details.

Definition 4.15. *Let X, Y be Banach spaces, $1 < p < \infty$. A function $\mathcal{M} \in C^\infty(\mathbb{R};\mathcal{B}(X,Y))$ is an $L^p_{X,Y}-$multiplier if there exists a bounded operator $T : L^p(\mathbb{R};X) \to L^p(\mathbb{R};Y)$ such that for all $f \in \mathcal{F}^{-1}\mathcal{D}(\mathbb{R};X)$*

$$Tf \in \mathcal{S}(\mathbb{R};Y), \quad \textit{and} \quad (Tf)^\wedge(s) = \mathcal{M}(s)\hat{f}(s), \quad s\in\mathbb{R}.$$

Definition 4.16. *A family of operators $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called $\mathcal{R}$-bounded if there is a constant $C > 0$ such that for all $T_1, ..., T_n \in \mathcal{T}$, $x_1, ..., x_n \in X$, $n \in \mathbb{N}$,*

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t) T_j x_j \right\|_Y dt \leq C \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|_X dt, \tag{4.14}$$

where (r_j) is a sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$, e.g. the Rademacher functions $r_j(t) = \operatorname{sgn}(\sin(2^j \pi t))$. The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ and we denote it by $R_p(\mathcal{T})$.

We note that in a Hilbert space every normbounded set $\mathcal{T}$ is $\mathcal{R}$-bounded. Several properties of R-bounded families can be founded in [22]. For the reader's convenience, we summarize here from [22, Section 3] some results.

(a) If $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded then it is uniformly bounded, and

$$\sup\{||T|| : T \in \mathcal{T}\} \leq R_p(\mathcal{T}).$$

(b) The definition of R-boundedness is independent of $p \in [1, \infty)$.

(c) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X,Y)$ is R-bounded if and only if $\mathcal{T}$ is uniformly bounded.

(d) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X,Y)$ be R-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \leq R_p(\mathcal{T}) + R_p(\mathcal{S})$.

(e) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$ be R-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is R-bounded, and $R_p(\mathcal{S}\mathcal{T}) \leq R_p(\mathcal{S}) R_p(\mathcal{T})$.

(g) Let X, Y be Banach spaces and $\mathcal{T} \subset \mathcal{B}(X,Y)$ be R-bounded. If $\{\alpha_k\}_{k \in \mathbb{Z}}$ is a bounded sequence, then $\{\alpha_k T : T \in \mathcal{T}\}$ is R-bounded.

The following operator-valued multiplier theorem is due to Weis [34, Theorem 3.4].

Theorem 4.17. *Let X, Y be UMD-spaces and $1 < p < \infty$. Suppose that $\mathcal{M} \in C^1(\mathbb{R}; \mathcal{B}(X,Y))$, and that the sets*

$$\{\mathcal{M}(s) : s \in \mathbb{R}\} \quad \text{and} \quad \{s\mathcal{M}'(s) : s \in \mathbb{R}\},$$

are $\mathcal{R}$-bounded. Then $\mathcal{M}$ is an $L^p_{X,Y}-$multiplier.

We recall that a Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then for all) $p \in (1, \infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f.$$

These spaces are also called $\mathcal{HT}$ spaces. It is a well known that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. This has been shown by Bourgain [18] and Burkholder [19]. Some examples of UMD-spaces include the Hilbert spaces, Sobolev spaces $W^s_p(\Omega)$, $1 < p < \infty$, Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, when X is a UMD-space. Moreover, a UMD-space is reflexive and therefore, $L^1(\Omega, \mu), L^\infty(\Omega, \mu)$ (in the case infinite dimensional) and $C^s([0, 2\pi]; X)$ are not UMD. More information on UMD spaces can be found in [18, 19].

Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ closed linear operators defined on a Banach space X satisfying $D(A) \cap D(M) \neq \{0\}$. We consider the following fractional differential equation

$$D^\alpha(Mu)(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{4.15}$$

where $1 \leq \alpha \leq 2$ and $f \in L^p(\mathbb{R}; X)$.

Definition 4.18. *Let $1 < p < \infty$. For $f \in L^p(\mathbb{R}; X)$, we call $u \in L^p(\mathbb{R}; X)$ a solution of equation (4.15) if $u \in W^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; [D(A) \cap D(M)])$ and u satisfies equation (4.15) for a.e. $t \in \mathbb{R}$.*

Definition 4.19. *We say that the equation (4.15) has maximal L^p-regularity if for each $f \in L^p(\mathbb{R}; X)$ there exists a unique solution u of equation (4.15).*

Remark 4.20.

Observe that if equation (4.15) has maximal L^p-regularity, it follows from the closed graph theorem that the map $L : L^p(\mathbb{R}; X) \to W^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; [D(A) \cap D(M)])$, which associates to f the unique solution u of equation (4.15) is linear and continuous.

Indeed, since A is a closed operator, we have that the space $H := W^{\alpha,p}(\mathbb{R}; X) \cap L^p(\mathbb{R}; [D(A) \cap D(M)])$ endowed with the norm

$$||u||_H := \|D^\alpha(Mu)\|_{L^p} + \|Au\|_{L^p} + \|u\|_{L^p}$$

is a Banach space.

Proposition 4.21. *Let $1 < p < \infty$, and $f \in \mathcal{F}^{-1}(\mathbb{R}; X)$, and $u \in L^p(\mathbb{R}; [D(A) \cap D(M)])$. Assume that $(is)^\alpha \in \rho_M(A)$ for all $s \in \mathbb{R}$. The following assertions are equivalent.*

(i) $u \in W^{\alpha,p}(\mathbb{R}; X)$ and u is a solution of equation (4.15);
(ii) $u \in \mathcal{S}(\mathbb{R}; [D(A) \cap D(M)])$ and $\hat{u}(s) = ((is)^\alpha M - A)^{-1}\hat{f}(s)$ for $s \in \mathbb{R}$.

Proof. $(ii) \Rightarrow (i)$. Observe that $\widehat{D^\alpha(Mu)}(s) = (is)^\alpha M\hat{u}(s)$, for all $s \in \mathbb{R}$. In fact, since $D^\alpha_+(e^{-ist}) = (is)^\alpha e^{-ist}$ for all $s \in \mathbb{R}$, we obtain by Lemma 2.1 that

$$(is)^\alpha M\hat{u}(s) = \int_{\mathbb{R}} (is)^\alpha e^{-ist} Mu(t)dt = \int_{\mathbb{R}} e^{-ist} D^\alpha(Mu)(t)dt = \widehat{D^\alpha Mu}(s).$$

Since A is a closed operator we get $\widehat{Au}(s) = A\hat{u}(s)$ and $(D^\alpha Mu - Au)\hat{}(s) = ((is)^\alpha M - A)\hat{u}(s) = \hat{f}(s)$, for all $s \in \mathbb{R}$. We obtain $D^\alpha(Mu) - Au = f$.

$(i) \Rightarrow (ii)$. Let $u \in L^p(\mathbb{R}; [D(A) \cap D(M)]) \cap W^{\alpha,p}(\mathbb{R}; X)$ be a solution of equation (4.15). Take $\phi \in \mathcal{S}(\mathbb{R})$ and define $R_M(A, s) := ((is)^\alpha M - A)^{-1}$, $s \in \mathbb{R}$. From Lemma 2.1 and Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \phi(s)\, R_M(A, s)\hat{f}(s)ds &= \int_{\mathbb{R}} \phi(s)\, R_M(A, s) \int_{\mathbb{R}} e^{-ist} f(t)dtds \\
&= \int_{\mathbb{R}} \phi(s)\, R_M(A, s) \int_{\mathbb{R}} e^{-ist}[D^\alpha(Mu)(t) - Au(t)]dtds \\
&= \int_{\mathbb{R}} \phi(s)\, R_M(A, s) \int_{\mathbb{R}} (Mu)(t)D^\alpha_+(e^{-ist}) - e^{-ist} Au(t)dtds \\
&= \int_{\mathbb{R}} \phi(s)\, R_M(A, s) \int_{\mathbb{R}} ((is)^\alpha M - A)u(t)e^{-ist}dtds \\
&= \int_{\mathbb{R}} \phi(s) \int_{\mathbb{R}} u(t)e^{-ist}dtds \\
&= \int_{\mathbb{R}} u(t) \int_{\mathbb{R}} \phi(s)e^{-ist}dsdt \\
&= \int_{\mathbb{R}} u(t)\hat{\phi}(t)dt.
\end{aligned}$$

Therefore, identifying $L^p(\mathbb{R}; [D(A) \cap D(M)])$ with a subspace of $\mathcal{S}'(\mathbb{R}; [D(A) \cap D(M)])$ by letting

$$\langle v, \phi\rangle = \int_{\mathbb{R}} v(t)\phi(t)dt,$$

for $v \in L^p(\mathbb{R}; [D(A) \cap D(M)])$ and $\phi \in \mathcal{S}(\mathbb{R})$. From the identity above, we have that $\hat{u}(s) = ((is)^\alpha M - A)^{-1}\hat{f}(s)$, for all $s \in \mathbb{R}$, and $\mathcal{F}u = ((i\cdot)^\alpha M - A)^{-1}\hat{f}(\cdot) \in \mathcal{D}(\mathbb{R}; [D(A) \cap D(M)])$. Hence $u \in \mathcal{S}(\mathbb{R}; [D(A) \cap D(M)])$. □

For $\beta > 0$ we define the following weighted L^p and Sobolev spaces on $\mathbb{R}$ with values in the Banach spaces X

$$L^p_\beta(\mathbb{R}; X) := \{f : \mathbb{R} \to X \text{ measurable } : \|f\|_{\beta,p} < \infty\},$$

$$W^{\alpha,p}_\beta(\mathbb{R}; X) := \{f : \mathbb{R} \to X \text{ measurable } : f, f', \dots, f^n \in L^p_\beta(\mathbb{R}; X)\} \text{ with } n = \lceil\alpha\rceil,$$

where $\|f\|_{\beta,p} := \left(\int_{\mathbb{R}} \|e^{-\beta|t|}f(t)\|^p dt\right)^{1/p}$ is the norm in $L^p_\beta(\mathbb{R};X)$ and $\|f\|_{\beta,p} + \|f'\|_{\beta,p} + \cdots + \|f^{(n)}\|_{\beta,p}$ is the norm in $W^{\alpha,p}_\beta(\mathbb{R};X)$.

As in Definition 4.18, for $f \in L^p_\beta(\mathbb{R};X)$ we call $u \in L^p_\beta(\mathbb{R};X)$ a *solution* of equation (4.15) if $u \in W^{\alpha,p}_\beta(\mathbb{R};X) \cap L^p_\beta(\mathbb{R};[D(A)\cap D(M)])$ and u satisfies equation (4.15) for a.e. $t \in \mathbb{R}$.

Further, we define the following mapping

$$\begin{array}{rrcl} ^{-}: & L^p_\beta(\mathbb{R};X) & \to & L^p(\mathbb{R};X) \\ & u & \mapsto & \bar{u}, \text{ where } \bar{u}(t) := e^{-\beta|t|}u(t). \end{array}$$

The function $^{-}$ is an isomorphism between $L^p_\beta(\mathbb{R};X)$ and $L^p(\mathbb{R};X)$.

Lemma 4.22. [28] *If $\alpha, \beta > 0$, then*

$$D^\alpha(e^{-\beta|t|}f(t)) = e^{-\beta|t|}\sum_{k=0}^{\infty}\binom{\alpha}{k}(-\mathrm{sgn}(t)\beta)^k D^{\alpha-k}f(t), \quad t \in \mathbb{R},$$

where $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)}{k!}$.

The following lemma establishes a connection between solutions in $L^p(\mathbb{R};X)$ and solutions in $L^p_\beta(\mathbb{R};X)$.

Lemma 4.23. [28] *Let $1 < p < \infty$, $\beta > 0$ and $f \in L^p_\beta(\mathbb{R};X)$. Then $u \in W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)\cap D(M)])$ is solution of equation (4.15) if and only if $\bar{u} \in W^{\alpha,p}(\mathbb{R};X)\cap L^p(\mathbb{R};[D(A)\cap D(M)])$ is solution of*

$$D^\alpha(M\bar{u})(t) = A\bar{u}(t) + \bar{f}(t) + e^{-\beta|t|}\sum_{k=1}^{\infty}\binom{\alpha}{k}(-\mathrm{sgn}(t)\beta)^k D^{\alpha-k}(e^{\beta|t|}\bar{u}(t)). \tag{4.16}$$

We notice that this result includes the cases of first and second order treated in [32] (for $M = I$). In fact, let $f \in L^p_\beta(\mathbb{R};X)$. If $\alpha = 1$, then $u \in L^p_\beta(\mathbb{R};X)$ is solution of $u'(t) = Au(t) + f(t)$ if and only if $\bar{u} \in L^p(\mathbb{R};X)$ is solution of

$$\bar{u}'(t) = A\bar{u}(t) + \bar{f}(t) - \beta\,\mathrm{sgn}(t)\,\bar{u}(t).$$

and if $\alpha = 2$, then it follows that $u \in L^p_\beta(\mathbb{R};X)$ is solution of $u''(t) = Au(t) + f(t)$ if and only if $\bar{u} \in L^p(\mathbb{R};X)$ is solution of

$$\bar{u}''(t) = A\bar{u}(t) + \bar{f}(t) - \beta^2\,\bar{u}(t) - 2\,\mathrm{sgn}(t)\,\bar{u}'(t).$$

See [32, Chapter 3] and [32, Chapter 6] for $\alpha = 1$ and $\alpha = 2$, respectively.

Lemma 4.24. *Assume that the operators A and M commute. If the equation (4.15) has maximal L^p-regularity, then there exists $\beta > 0$ such that for all $f \in L^p_\beta(\mathbb{R};X)$ there exists an unique solution $u \in W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)\cap(M)])$ of equation (4.15) and the solution operator $\mathcal{L}_\beta : L^p_\beta(\mathbb{R};X) \to W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)\cap D(M)])$ is bounded.*

Proof. Let $f \in L^p_\beta(\mathbb{R};X)$. From Lemma 4.23 we obtain that $u \in W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)\cap D(M)])$ is solution of equation (4.15) if and only if $\bar{u} \in W^{\alpha,p}(\mathbb{R};X)\cap L^p(\mathbb{R};[D(A)\cap D(M)])$ is solution of equation (4.16). Define the mapping $T_\beta : W^{\alpha,p}(\mathbb{R};X) \to W^{\alpha,p}(\mathbb{R};X)\cap L^p(\mathbb{R};[D(A)\cap D(M)])$ by

$$T_\beta g := L\left(-h_g\right),$$

where L is the solution operator of equation (4.15) and for a given function g, h_g denotes the function

$$h_g(t) = e^{-\beta|t|}\sum_{k=1}^{\infty}\binom{\alpha}{k}(-\mathrm{sgn}(t)\beta)^k\beta^{-1}D^{\alpha-k}(e^{\beta|t|}g(t)).$$

From Lemma 4.22 in follows that $h_g \in L^p(\mathbb{R};X)$, hence T_β is well-defined and is a bounded operator. Moreover, $MT_\beta = T_\beta M$. In fact, because M is a closed operator, an easy computation shows that $h_{Mg} = Mh_g$. Moreover, if $u = L(f)$ and $v = L(Mf)$ then $D^\alpha(Mu) = Au + f$ and $D^\alpha(Mv) = Av + Mf$. Therefore $MD^\alpha(Mu) = AMu + Mf$ and since M is a closed operator, $D^\alpha(MMu) = MAu + Mf = AMu + Mf$, that is, $L(Mf) = Mu$. Since $L(Mf) = v$, the uniqueness implies that $Mu = v$. We conclude that $ML(f) = Mu = v = LM(f)$, which means that M and L commute. Now, observe that

$$MT_\beta g = ML(-h_g) = LM(-h_g) = L(-Mh_g) = L(-h_{Mg}) = T_\beta(Mg).$$

On the other hand, by (4.16) we have

$$\begin{aligned}
D^\alpha((1+\beta T_\beta)(M\bar{u}))(t) &= D^\alpha(M\bar{u})(t) + \beta D^\alpha(T_\beta M\bar{u})(t)\\
&= A\bar{u}(t) + \bar{f}(t) + \beta h_{\bar{u}}(t) + \beta D^\alpha(T_\beta M\bar{u})(t)\\
&= A\bar{u}(t) + \bar{f}(t) + \beta h_{\bar{u}}(t) + \beta D^\alpha(M(T_\beta\bar{u}))(t)\\
&= A\bar{u}(t) + \bar{f}(t) + \beta h_{\bar{u}}(t) + \beta D^\alpha(ML(-h_{\bar{u}}))(t)\\
&= A\bar{u}(t) + \bar{f}(t) + \beta h_{\bar{u}}(t) + \beta[AL(-h_{\bar{u}})(t) - h_{\bar{u}}(t)]\\
&= A\bar{u}(t) + \bar{f}(t) + \beta[AL(-h_{\bar{u}})(t)]\\
&= A[\bar{u}(t) + \beta L(-h_{\bar{u}})(t)] + \bar{f}(t)\\
&= A(1+\beta T_\beta)\bar{u}(t) + \bar{f}(t).
\end{aligned}$$

Therefore, $L(\bar{f}) = (1+\beta T_\beta)\bar{u}$. If β is small enough, then $(1+\beta T_\beta)$ is invertible. For this β, we get that

$$\mathcal{M}_\beta f = (^-)^{-1}(1+\beta T_\beta)^{-1}L(\bar{f}),$$

and by the closed graph theorem, the operator $\mathcal{M}_\beta$ which takes $f \in L^p_\beta(\mathbb{R};X)$ into the unique solution $u \in W^{\alpha,p}_\beta(\mathbb{R};X) \cap L^p_\beta(\mathbb{R};[D(A)\cap D(M)])$ of equation (4.15) is a bounded operator. □

The main result in this section is the following theorem.

Theorem 4.25. *Assume that X is a UMD-space and $1 < p < \infty$. If the operators A and M commute, then following assertions are equivalent.*

(i) Equation (4.15) has maximal L^p-regularity;

(ii) $(is)^\alpha \in \rho_M(A)$ for all $s \in \mathbb{R}$ and the set $\{(is)^\alpha M((is)^\alpha M - A)^{-1}\}_{s\in\mathbb{R}}$ is $\mathcal{R}$-bounded.

Proof. $(i) \Rightarrow (ii)$. Assume that equation (4.15) has maximal L^p-regularity. Let $s \in \mathbb{R}$ and suppose

$$((is)^\alpha M - A)x = 0, \tag{4.17}$$

for $x \in D(A)\cap D(M)$. Let $u(t) := e^{ist}x$. Then $u \in W^{\alpha,p}_\beta(\mathbb{R};X) \cap L^p_\beta(\mathbb{R};[D(A)\cap D(M)])$ for all $\beta > 0$. Observe that u is a solution to equation (4.15) with $f \equiv 0$. In fact, $D^\alpha u(t) = (is)^\alpha e^{ist}x$ (see [27, p. 248]). Moreover, by (4.17) we have

$$Au(t) = e^{ist}Ax = e^{ist}(is)^\alpha Mx = D^\alpha u(t).$$

Choosing the number $\beta > 0$ given in Lemma 4.24, we obtain by uniqueness that $u \equiv 0$, that is, $x = 0$. Hence $((is)^\alpha M - A)$ is injective.

Now, we prove the surjectivity. Let $y \in X$ be arbitrary. Let $s \in \mathbb{R}$ and β be small enough as in Lemma 4.24. Let f_s defined by $f_s(t) := e^{ist}y$. Clearly $f_s \in L^p_\beta(\mathbb{R};X)$. Let $\mathcal{M}_\beta : L^p_\beta(\mathbb{R};X) \to W^{\alpha,p}_\beta(\mathbb{R};X) \cap L^p_\beta(\mathbb{R};[D(A)\cap D(M)])$ be the bounded operator which takes each $f \in L^p_\beta(\mathbb{R};X)$ to the unique solution u of equation (4.15).

Let $u = \mathcal{M}_\beta f_s$. For fixed $r \in \mathbb{R}$ we have that $v_1(t) := u(t+r)$ and $v_2(t) := e^{isr}u(t)$ are both solutions of (4.15) with $g(t) = e^{isr}f_s(t)$. Hence, $v_1 = v_2$, that is, $u(t+r) = e^{isr}u(t)$ for all $r,t \in \mathbb{R}$. Let $x = u(0) \in D(A)\cap D(M)$. With $r = -t$ we obtain $u(t) = e^{ist}x$ for all $t \in \mathbb{R}$. Since $D^\alpha u(t) = (is)^\alpha e^{ist}x$ we have $D^\alpha u(0) = (is)^\alpha x$ and therefore,

$$((is)^\alpha M - A)x = D^\alpha u(0) - Au(0).$$

Since $u(t)$ satisfies the equation (4.15) for all $t \in \mathbb{R}$, we obtain,

$$((is)^\alpha M - A)x = D^\alpha u(0) - Au(0) = f_s(0) = y, \tag{4.18}$$

which means that $((is)^\alpha M - A)$ is surjective for all $s \in \mathbb{R}$. Moreover, by (4.18) we obtain that for all $y \in X$,

$$\|((is)^\alpha M - A)^{-1}y\| = \|x\| = \|u(0)\| = \|\mathcal{M}_\beta f_s(0)\| \le \|\mathcal{M}_\beta\|\|f_s(0)\| = \|\mathcal{M}_\beta\|\|y\|,$$

that is, $((is)^\alpha M - A)^{-1}$ is a bounded operator. We conclude that $(is)^\alpha \in \rho_M(A)$ for all $s \in \mathbb{R}$.

Now, we shall prove that $\{(is)^\alpha M((is)^\alpha M-A)^{-1}: s\in\mathbb{R}\}$ is an $\mathcal{R}$-bounded set. Since the solution operator L of equation (4.15) is bounded, we have that if $f\in\mathcal{F}^{-1}(\mathbb{R};X)$ then $u=Lf\in\mathcal{S}(\mathbb{R};[D(A)\cap D(M)])$ (see Proposition 4.21) and $\hat{u}(s)=\widehat{Lf}(s)=((is)^\alpha M-A)^{-1}\hat{f}(s)$, for all $s\in\mathbb{R}$. Therefore, the function $N:\mathbb{R}\to\mathcal{B}(X;[D(A)\cap D(M)])$ given by $N(s)=((is)^\alpha M-A)^{-1}$ defines an $L^p_{X;[D(A)\cap D(M)]}-$multiplier. We get that $\{N(s): s\in\mathbb{R}\}$ is $\mathcal{R}$-bounded (see [21, Proposition 1]). Since $A: D(A)\to X$ is an isomorphism we obtain that $\{AN(s): s\in\mathbb{R}\}$ $\mathcal{R}$-bounded. The identity $(is)^\alpha MN(s)=I+AN(s)$ implies that the set $\{(is)^\alpha M((is)^\alpha M-A)^{-1}: s\in\mathbb{R}\}$ is $\mathcal{R}$-bounded.

$(ii)\Rightarrow(i)$. Define the operator $N(s):=((is)^\alpha M-A)^{-1}$, where $s\in\mathbb{R}$. By hypothesis $N\in C^1(\mathbb{R};\mathcal{B}(X,[D(A)\cap D(M)]))$. We claim that $\{N(s): s\in\mathbb{R}\}$ is a $L^p_{X,[D(A)\cap D(M)]}-$multiplier.

In fact, the hypothesis and the identity $(is)^\alpha MN(s)-I=AN(s)$ show that $\{N(s): s\in\mathbb{R}\}$ is $\mathcal{R}$-bounded. Since

$$sN'(s)=-\alpha(is)^\alpha MN(s)N(s),$$

for all $s\in\mathbb{R}$, we get as consequence that $\{sN'(s): s\in\mathbb{R}\}$ is $\mathcal{R}$-bounded. By Theorem 4.17, $\{N(s): s\in\mathbb{R}\}$ is an $L^p_{X,[D(A)\cap D(M)]}-$multiplier. Thus, there exits a bounded operator

$$T: L^p(\mathbb{R};X)\to L^p(\mathbb{R};[D(A)\cap D(M)])$$

such that for $f\in\mathcal{F}^{-1}\mathcal{D}(\mathbb{R};X)$, $u:=Tf\in\mathcal{S}(\mathbb{R};[D(A)\cap D(M)])$ and $\hat{u}(s)=((is)^\alpha M-A)^{-1}\hat{f}(s)$ for all $s\in\mathbb{R}$. The Proposition 4.21 implies that u is a solution of equation (4.15). Observe that,

$$\|u\|_{L^p(\mathbb{R};[D(A)\cap D(M)])}\le\|T\|\,\|f\|_{L^p(\mathbb{R};X)}.$$

Now, let $f\in L^p(\mathbb{R};X)$ be an arbitrary function. Then there exist $f_n\in\mathcal{F}^{-1}\mathcal{D}(\mathbb{R};X)$ such that $f_n\to f$ in $L^p(\mathbb{R};X)$. Let $u_n=Tf_n$. Then u_n is a solution of equation (4.15) for f_n. Moreover $u_n\to u:=Tf$ in $L^p(\mathbb{R};[D(A)\cap D(M)])$. By Lemma 2.1 we get for each $\phi\in\mathcal{D}(\mathbb{R})$ that

$$\int_{\mathbb{R}}(Au_n(t)+f_n(t))\phi(t)dt=\int_{\mathbb{R}}D^\alpha(Mu_n)(t)\phi(t)dt=\int_{\mathbb{R}}Mu_n(t)D^\alpha_+\phi(t)dt.$$

If $n\to\infty$ then by Lemma 2.1 we obtain that u is a weak solution of equation (4.15) and therefore $D^\alpha(Mu)=Au+f$, that is, the equation (4.15) has the maximal L^p-regularity property.

To see the uniqueness, suppose that

$$D^\alpha(Mu)(t)=Au(t),\quad t\in\mathbb{R}, \tag{4.19}$$

with $u\in W^{\alpha,p}(\mathbb{R};X)\cap L^p(\mathbb{R};[D(A)\cap D(M)])$.

A easy computation shows that the Carleman transform of fractional derivative of u satisfies

$$\widetilde{D^\alpha u}(\lambda)=\lambda^\alpha\tilde{u}(\lambda)-\sum_{k=0}^{n-1}u^{(k)}(0)\lambda^{\alpha-1-k},\ \text{for } \mathrm{Re}\lambda\neq0,\ n=\lceil\alpha\rceil.$$

Taking Carleman transform in (4.19), we get

$$(\lambda^\alpha M-A)\tilde{u}(\lambda)=\sum_{k=0}^{n-1}u^{(k)}(0)\lambda^{\alpha-1-k},\ \text{for } \mathrm{Re}\lambda\neq0,\ n=\lceil\alpha\rceil.$$

Since $(is)^\alpha\in\rho_M(A)$ for all $s\in\mathbb{R}$, it follows that the Carleman spectrum $\tilde{u}$ of u is empty and therefore $u=0$ (see [3, Theorem 4.8.2]).

□

Corollary 4.26. *Let H be Hilbert space and let $A: D(A)\subseteq H\to H$ $M: D(M)\subset H\to H$ closed linear operators such that A and M commute. Then, the following assertions are equivalent for $1<p<\infty$.*

(i) Equation (4.15) has maximal L^p-regular;

(ii) $(is)^\alpha\in\rho_M(A)$ for all $s\in\mathbb{R}$ and $\sup_{s\in\mathbb{R}}\|(is)^\alpha M((is)^\alpha M-A)^{-1}\|<\infty$.

Corollary 4.27. *In the context of Theorem 4.25, if condition (ii) is fulfilled, we have that $D^\alpha Mu$, $Au\in L^p(\mathbb{R};X)$. Moreover, there exists a constant $C>0$ independent of $f\in L^p(\mathbb{R};X)$ such that*

$$\|D^\alpha(Mu)\|_{L^p(\mathbb{R};X)}+\|Au\|_{L^p(\mathbb{R};X)}\le C\|f\|_{L^p(\mathbb{R};X)}. \tag{4.20}$$

For $y \in X$ and $r \in \mathbb{R}$, we define $f_r(t) := e^{irt}y$. Is clear that $f_r \in L^p_\beta(\mathbb{R};X)$ for all $r \in \mathbb{R}$ since

$$||f_r||_{\beta,p} = \left(\int_{\mathbb{R}} e^{-\beta|t|p}dt\right)^{1/p} ||y|| =: C_{\beta,p}\, ||y|| .$$

Theorem 4.28. *Let A be a linear operator on a Banach space X. Assume that equation (4.15) has maximal L^p-regularity for equation (4.15) for some $p \in (1,\infty)$. Then $(is)^\alpha \in \rho_M(A)$ for all $s \in \mathbb{R}$ and there exists a constant $C > 0$ such that*

$$\|((is)^\alpha M - A)^{-1}\| \leq \frac{C}{1+|s|^\alpha}\,, \quad s \in \mathbb{R}.$$

Proof. As in the proof of Theorem 4.25 we have that $(is)^\alpha \in \rho_M(A)$ for all $s \in \mathbb{R}$. Let $s \in \mathbb{R}$ and $y \in X$. Since $((is)^\alpha M - A)$ is bijective, there exists $z \in D(A)$ such that

$$((is)^\alpha M - A)z = y.$$

Let β be small enought as in Lemma 4.24. From proof of Theorem 4.25 we have that for $f_s(t) := e^{ist}y$, the unique solution of equation (4.15) is $u_s(t) := e^{ist}z$. Moreover,

$$\|u_s\|_{\beta,p} = C_{\beta,p}\,\|z\|.$$

Let $n = \lceil \alpha \rceil$. Observe that

$$\begin{aligned}
\|u_s\|_{\beta,p} &= C_{\beta,p}\,\|z\| \\
\|u_s'\|_{\beta,p} &= |s|C_{\beta,p}\,\|z\| \\
\|u_s''\|_{\beta,p} &= |s|^2C_{\beta,p}\,\|z\| \\
&\vdots \\
\|u_s^{(n)}\|_{\beta,p} &= |s|^nC_{\beta,p}\,\|z\|
\end{aligned}$$

By Lemma 4.24 we have

$$\begin{aligned}
(1+|s|+|s|^2+...+|s|^n)C_{\beta,p}\|z\| &= \|u_s\|_{\beta,p} + \|u_s'\|_{\beta,p} + ... + \|u_s^{(n)}\|_{\beta,p} \\
&= \|u_s\|_{W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)\cap D(M)])} \\
&= \|\mathcal{M}_\beta f_s\|_{W^{\alpha,p}_\beta(\mathbb{R};X)\cap L^p_\beta(\mathbb{R};[D(A)\cap D(M)])} \\
&\leq \|\mathcal{M}_\beta\|\|f_s\|_{L^{\alpha,p}_\beta(\mathbb{R};X)} \\
&= CC_{\beta,p}\|y\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|((is)^\alpha M - A)^{-1}y\| &\leq \frac{C}{1+|s|+|s|^2+...+|s|^n}\|y\| \\
&\leq \frac{C}{1+|s|^\alpha}\|y\|.
\end{aligned}$$

□

5. Examples

Example 5.29.

Let $1/2 < \alpha \leq 1$. We consider the problem

$$\frac{\partial(m(x)u)}{\partial t} - \Delta u = f(t,x), \text{ in } \mathbb{R}\times\Omega \tag{5.21}$$

$$u = 0, \text{ in } \mathbb{R}\times\partial\Omega, \tag{5.22}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m(x) \geq 0$ is a given measurable bounded function on Ω and f is a function on $\mathbb{R}\times\Omega$.

Let M be the multiplication operator by m. We notice that if m vanishes in a measurable subset of Ω, then M^{-1} is an unbounded operator. If we take $X = H^{-1}(\Omega)$ then by [6, p.38] (see also references therein), we have that there exists a constant $c > 0$ such that

$$||M(zM - \Delta)^{-1}|| \leq \frac{c}{1 + |z|},$$

whenever $\mathrm{Re} z \geq -c(1 + |\mathrm{Im}(z)|)$. In particular, if $z = (is)^\alpha = |s|^\alpha e^{\frac{\pi\alpha i}{2}\mathrm{sgn}(s)}$, that is, $z = (is)^\alpha = s^\alpha \left(\cos\left(\frac{\pi\alpha}{2}\right) + i\sin\left(\frac{\pi\alpha}{2}\right)\right)$ is $s \geq 0$ and $z = (is)^\alpha = |s|^\alpha \left(\cos\left(\frac{\pi\alpha}{2}\right) - i\sin\left(\frac{\pi\alpha}{2}\right)\right)$ for $s < 0$. Since $/2 < \alpha \leq 1$ we obtain that $\cos\left(\frac{\pi\alpha}{2}\right) \geq 0$ and $\sin\left(\frac{\pi\alpha}{2}\right) \geq 0$, obtaining that

$$||M((is)^\alpha M - \Delta)^{-1}|| \leq \frac{c}{1 + |s|^\alpha},$$

for all $s \in \mathbb{R}$.

imaginary axis we have $||M(itM - \Delta)^{-1}|| \leq \frac{c}{1+|t|}$, for all $t \in \mathbb{R}$. Therefore, we conclude by Theorem 4.25 that the equation (5.21) is C^α-well posed. Thus, given $f \in C^\alpha(\mathbb{R} \times \Omega; X)$ there exists a unique solution u to problem (5.21) which satisfy $(m(x)u)', \Delta u \in C^\alpha(\mathbb{R} \times \Omega; X)$.

Example 5.30.

Let P be a densely defined positive selfadjoint operator defined on a Hilbert space X with $P \geq \delta > 0$. Let $M = P - \varepsilon$ with $\varepsilon \leq \delta$, and let $A = -\sum_{i=0}^k a_i P^i$ with $a_i \geq 0$, $a_k > 0$, and $k \geq 2$ is an integer. From [23, p. 73] we have that there exists a constant $c > 0$ such that

$$||M(zM - A)^{-1}|| \leq \frac{c}{1 + |z|},$$

whenever $\mathrm{Re} z \geq -c(1 + |\mathrm{Im}(z)|)$. Thus, in the imaginary axis we have $||M(itM - A)^{-1}|| \leq \frac{c}{1+|t|}$, for all $t \in \mathbb{R}$. Hence, in this conditions the equation (4.15) is C^α-well posed.

Example 5.31.

For $(x, t) \in \Omega \times \mathbb{R}$ where $\Omega = (0, 1)$, consider the problem

$$\frac{\partial}{\partial t}\Big(1 - \frac{\partial^2}{\partial x^2}\Big)u(x,t) = -\frac{\partial^4}{\partial x^4}u(x,t) + f(x,t) \tag{5.23}$$

$$u = 0, \quad \text{in } \partial\Omega \times \mathbb{R}. \tag{5.24}$$

In the space $X = L^2(\Omega)$, let $P = -\frac{\partial^2}{\partial x^2}$, with domain $D(P) = H^2(\Omega) \cap H_0^1(\Omega)$. Observe that P is a positive selfadjoint operator in X. If $M = P + I$, and $A = -P^2$, then the equation (5.23) can be written in the form of (4.15). By Example 5.30, the equation (5.23) is C^α-well posed.

Example 5.32.

Consider the problem

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2}{\partial x^2} + 1\Big)u(t,x) = -a\frac{\partial^2}{\partial x^2}u(t,x) - ku(t,x) + f(t,x), \quad t \in \mathbb{R}, x \in [0,\pi] \tag{5.25}$$

$$u(t,0) = u(t,\pi) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,\pi) = 0, \quad t \in \mathbb{R} \tag{5.26}$$

where a is positive constant and $-2a < k < 4a$. In $X = C_0([0,\pi]) = \{u \in C([0,\pi]) : u(0) = u(\pi)\}$ take K the realization of $\frac{\partial^2}{\partial x^2}$ with domain

$$D(K) = \{u \in C^2([0,\pi]) : u(0) = u(\pi) = \frac{\partial^2}{\partial x^2}u(0) = \frac{\partial^2}{\partial x^2}u(\pi) = 0\}.$$

If $M = K + I$, and $A = aM + (k - a)I$, then the equation (5.25) can be written in the form of (4.15). By [6, p.39] or [23] we have, as in the above example:

$$||M(itM - A)^{-1}|| \leq \frac{c}{1 + |t|}$$

for all $t \in \mathbb{R}$. Therefore, by Theorem 4.25 the equation (5.25) is C^α-well posed, that is, for all $f \in C^\alpha(\mathbb{R} \times [0,\pi]; C_0([0,\pi]))$ there exists a unique solution u of (5.25) with maximal regularity $\frac{\partial^2 u}{\partial x^2} \in C^\alpha(\mathbb{R} \times [0,\pi]; C_0([0,\pi]))$.

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