SUBORDINATION PRINCIPLE FOR SUBDIFFUSION EQUATIONS WITH MEMORY.

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ABSTRACT. In this paper we study the existence of mild solutions to subdiffusion equations with memory

$$(*)\begin{cases} \partial_t^\alpha u(t) &= Au(t) + \displaystyle\int_0^t \kappa(t-s)Au(s)ds, \quad t \geq 0 \\ u(0) &= x, \end{cases}$$

where $0 < \alpha < 1$, A is a closed linear operator defined on a Banach space X, the initial value x belongs to X and κ is a suitable kernel in $L^1_{\text{loc}}(\mathbb{R}_+)$. First, we find a subordination formula for the solution operator of $(*)$ and then we study its connection with the existence of mild solution to the first order diffusion equation with memory.

1. Introduction

In the problem of the heat conduction, the classical model reads as

$$\begin{cases} u_t = \lambda\Delta u + f, & \text{in} \quad D := \Omega \times J \\ u(x,t) = 0, & \text{on} \quad \partial D \times J \\ u(x,0) = u_0(x), & \text{in} \quad \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $(N = 1, 2, 3)$, $J = (0, T]$, $T > 0$, f is a suitable function (known as the *forcing term*), $\lambda > 0$ is the thermal diffusion coefficient, Δ is de Laplacian operator defined in certain domain, $u_0(x)$ is the initial temperature at point $x \in \Omega$ and $\partial\Omega$ denotes the boundary of Ω. In homogeneous and isotropic media, this model gives a good prediction of the temperature $u(x,t)$ at time t in any x point of Ω. However, in some materials, such as materials with fading memory, particularly at low temperature, this model is not completely satisfactory. Moreover, in this model the thermal disturbance at any point is propagated instantly to everywhere of the domain, which is in general unrealistic. The theory of heat conduction in such materials was firstly studied by Gurtin and Pipkin [14] where the authors, after a linearization, arrived to the model given by the Volterra equation

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t a(t-s)Au(s)ds + f(t), \ t \geq 0 \\ u(0) = x, \end{cases} \tag{1.2}$$

where A is a closed operator (typically is the Laplacian operator), a is a locally integrable kernel known as the *heat relaxation function,* and f is a suitable continuous function. The mild solution to equation (1.2) it is well known: if A is the generator of a resolvent family $\{R^a(t)\}_{t\geq 0}$ (see [12]), then the solution u to (1.2) is given by the variation of constants formula

$$u(t) = R^a(t)x + \int_0^t R^a(t-s)f(s)ds, \quad t \geq 0, \tag{1.3}$$

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where the Laplace transform $\hat{\cdot}$ of $R^a(t)$ satisfies

$$\widehat{R^a}(\lambda) = \frac{1}{1+\hat{a}(\lambda)}\left(\frac{\lambda}{1+\hat{a}(\lambda)} - A\right)^{-1}$$

for all $\lambda \in \mathbb{C}$ such that $\frac{\lambda}{1+\hat{a}(\lambda)} \in \rho(A)$. The existence of mild solutions to equations with memory has been widely study in the last decades, see for instance [13, 17, 32, 31, 33] and the references therein. We notice that if $a(t) = 0$ for all $t \geq 0$, (that is, the heat equation without memory) then $R^a(t)$ corresponds to the C_0-semigroup generated by A.

In the last two decades, the theory of fractional differential equations has been a subject of great interest in many areas of mathematics, physics, mechanics, chemistry and biology, see for instance [18, 20, 24, 25]. In particular, recently the subdiffusion equation with memory in porous media

$$\left\{\begin{array}{l} \partial_t^\alpha u(t) = Au(t) + \displaystyle\int_0^t \kappa(t-s)Au(s)ds + f(t), \ t \geq 0 \\ u(0) = x, \end{array}\right. \tag{1.4}$$

where $\partial_t^\alpha u$ denotes the Caputo fractional derivative of u, $0 < \alpha < 1$, $x \in X$, and κ is suitable kernel has been studied in [21, 22] and [23]. Moreover, very recently, the authors in [1] have obtained interesting results on the study of the existence and uniqueness of mild solutions to (1.4) for $\kappa(t) = e^{-\rho t}\frac{t^{\mu-1}}{\Gamma(\mu)}$ where $\rho \geq 0$ and $0 < \mu \leq 1$.

On the other hand, if $a(t) = \kappa(t) = 0$ for all $t \geq 0$ in equations (1.2) and (1.4), then there exists an interesting relation (known as *subordination principle,* see for instance [2, 5, 6, 7]): if A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then A it also generates the α-times resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ given by

$$S_\alpha(t)x = \int_0^\infty \Phi_\alpha(r)T(rt^\alpha)xdr, \quad t \geq 0, x \in X, \tag{1.5}$$

where Φ_α is the Wright type function ([24, Appendix F])

$$\Phi_\alpha(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = \int_\gamma \mu^{\alpha-1}e^{\mu - z\mu^\alpha}d\mu,$$

where γ is a contour which starts and ends at $-\infty$ and encircles the origin once counterclockwise, see for instance [5, Chapter 3]. This interesting property implies (in case $a \equiv \kappa \equiv 0$) that if A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then the solution to (1.4) is given by

$$u(t) = S_\alpha(t)x + \int_0^t P_\alpha(t-s)f(s)ds,$$

where $\{S_\alpha(t)\}_{t\geq 0}$ is the fractional resolvent family given in (1.5) and $\{P_\alpha(t)\}_{t>0}$ is the family defined by (see [19, Theorem 3.1])

$$P_\alpha(t) = \alpha\int_0^\infty t^{\alpha-1}r\Phi_\alpha(r)T(rt^\alpha)dr, \quad t > 0.$$

The subordination principles are useful to study parabolic as well as hyperbolic problems, see for instance [29, Chapter I, Section 4]. In this paper, we study the problem of the existence of mild solutions to the subdiffusion equation with memory (1.4) and its connection with the existence of mild solution to the diffusion equation with memory (1.2) via a subordination principle. More precisely, in this paper we prove that if A is the generator of a resolvent family $\{R^a(t)\}_{t\geq 0}$ (associated to the equation (1.2)) then a subordination principle, allows to prove that A is also the generator of a fractional resolvent family $\{R_\alpha(t)\}_{t>0}$ and then, the solution to (1.4) can be written using a variation of parameters formula (as in (1.3)) in terms of $\{R_\alpha(t)\}_{t>0}$.

2. Resolvent families and A Subordination Principle

Given a Banach spaces $(X, \|\cdot\|)$, we denote by $\mathcal{B}(X)$ to the Banach space of all bounded and linear operators from X into X. For a closed linear operator A on X, $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$ is the resolvent operator of A which is defined for all $\lambda \in \rho(A)$.

A strongly continuous family of linear operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be *exponentially bounded* if there exist constants $M > 0$ and $w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{wt}$, for all $t > 0$.

Definition 2.1. *Let A be a closed and linear operator defined on a Banach space X and $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. We say that A is the generator of a* resolvent family, *if there exist $M > 0$, $\omega \geq 0$ and a strongly continuous function $R^a : [0, \infty) \to \mathcal{B}(X)$ such that $\|R^a(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, $\{\lambda : \mathrm{Re}\lambda > \omega\} \subset \rho_a(A)$ and for all $x \in X$,*

$$\frac{1}{1+\hat{a}(\lambda)}\left(\frac{\lambda}{1+\hat{a}(\lambda)} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} R^a(t)x dt, \ \ \mathrm{Re}\lambda > \omega,$$

where

$$\rho_a(A) := \left\{\mu \in \mathbb{C} : \left(\frac{\mu}{1+\hat{a}(\mu)} - A\right) \textit{ is invertible and } \left(\frac{\mu}{1+\hat{a}(\mu)} - A\right)^{-1} \textit{ is bounded}\right\}.$$

In this case, $\{R^a(t)\}_{t\geq 0}$ is called the resolvent family *generated by A.*

If A is the generator of a resolvent family $\{R^a(t)\}_{t\geq 0}$, $c(t) := 1$ and $b(t) := 1 + (1 * a)(t)$, then $\{R^a(t)\}_{t\geq 0}$ is a (b, c)-regularized family according to [27], and in particular, if $a \equiv 0$, then the resolvent family $\{R^a(t)\}_{t\geq 0}$ corresponds to the C_0-semigroup generated by A. It is a well-known fact that if A generates a resolvent family $\{R^a(t)\}_{t\geq 0}$, then solution u to (1.2) is given by the variation of parameters formula

$$u(t) = R^a(t)x + \int_0^t R^a(t-s)f(s)ds, \quad t \geq 0. \tag{2.6}$$

Given $\alpha > 0$, we define the function g_α as $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $\Gamma(\cdot)$ is the Gamma function. We note that if $\alpha, \beta > 0$, then $g_{\alpha+\beta} = g_\alpha * g_\beta$, where $(f * g)$ denotes the usual finite convolution $(f * g)(t) = \int_0^t f(t-s)g(s)ds$. For $0 < \alpha < 1$, the Caputo fractional derivative of order α of a function f is defined by

$$\partial_t^\alpha f(t) := (g_{1-\alpha} * f')(t) = \int_0^t g_{1-\alpha}(t-s)f'(s)ds.$$

An easy computation shows that if $\alpha = 1$, then $\partial_t^1 = \frac{d}{dt}$. For more details, examples and applications on fractional calculus, we refer to the reader to [24, 25].

For a locally integrable function $f : [0, \infty) \to X$, we define the *Laplace transform* of f, denoted by $\hat{f}(\lambda)$ (or $\mathcal{L}(f)(\lambda)$) as

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt,$$

provided the integral converges for some $\lambda \in \mathbb{C}$. An easy computation shows that $\hat{g}_\alpha(\lambda) = \frac{1}{\lambda^\alpha}$ for all $\mathrm{Re}(\lambda) > 0$ and applying the properties of the Laplace transform, it is easy to see that

$$\widehat{\partial_t^\alpha f}(\lambda) = \lambda^\alpha \hat{f}(\lambda) - \lambda^{\alpha-1}f(0) \tag{2.7}$$

for $0 < \alpha \leq 1$. Here, the power λ^α is uniquely defined by $\lambda^\alpha := |\lambda|^\alpha e^{i\arg(\lambda)}$, with $-\pi < \arg(\lambda) < \pi$.

For $\alpha, \beta > 0$ and $z \in \mathbb{C}$, the Mittag-Leffler function $E_{\alpha,\beta}$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

Given $\alpha > -1, \beta \in \mathbb{C}$ and $z \in \mathbb{C}$, the Wright function $W_{\alpha,\beta}$ is defined by

$$W_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \beta)}.$$

If $\beta \geq 0$, then for all $z \in \mathbb{C}$ and $\alpha > -1$, we have (see [24]) that

$$W_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{H_a} \mu^{-\beta} e^{\mu + z\mu^{-\alpha}} d\mu,$$

where H_a denotes the Hankel path defined as a contour that begins and $t = -\infty - ia$ $(a > 0)$, encircles the branch cut that lies along the negative real axis, and ends up at $t = -\infty + ib$ $(b > 0)$, see for instance [24].

Definition 2.2. [2, Definition 3.1] *For $0 < \alpha < 1$ and $\beta \geq 0$, we define the function $\psi_{\alpha,\beta}$ in two variables by*

$$\psi_{\alpha,\beta}(t,s) := t^{\beta-1} W_{-\alpha,\beta}(-st^{\alpha}), \quad t > 0, s \in \mathbb{C}.$$

By [2, Theorem 3.2] it follows that if $0 < \alpha < 1$ and $\beta \geq 0$, then $\psi_{\alpha,\beta}(t,s) \geq 0$ for $t, s > 0$ and

$$\int_0^{\infty} e^{-\lambda t} \psi_{\alpha,\beta}(t,s) dt = \lambda^{-\beta} e^{-\lambda^{\alpha} s}, \text{ for } s, \lambda > 0. \tag{2.8}$$

Definition 2.3. *Let A be a closed and linear operator defined on a Banach space X and $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$. Given $\alpha, \beta > 0$ we say that A is the generator of an (α,β)-resolvent family, if there exist $\omega \geq 0$ and a strongly continuous function $R^{\kappa}_{\alpha,\beta} : (0,\infty) \to \mathcal{B}(X)$ such that $R^{\kappa}_{\alpha,\beta}(t)$ is exponentially bounded, $\left\{\frac{\lambda^{\alpha}}{1+\hat{\kappa}(\lambda)} : \text{Re}\lambda > \omega\right\} \subset \rho(A)$, and for all $x \in X$,*

$$\frac{\lambda^{\alpha-\beta}}{1+\hat{\kappa}(\lambda)} \left(\frac{\lambda^{\alpha}}{1+\hat{\kappa}(\lambda)} - A\right)^{-1} x = \int_0^{\infty} e^{-\lambda t} R^{\kappa}_{\alpha,\beta}(t) x dt, \ \text{Re}\lambda > \omega. \tag{2.9}$$

In this case, $\{R^{\kappa}_{\alpha,\beta}(t)\}_{t>0}$ is called the (α,β)-resolvent family *generated by A.*

We notice that if $\alpha = \beta = 1$, then a $(1,1)$-resolvent family $\{R^{\kappa}_{1,1}(t)\}_{t>0}$ is a resolvent family according to Definition 2.1.

If $b(t) := g_{\alpha}(t) + (\kappa * g_{\alpha})(t)$ and A is the generator of an (α,β)-resolvent family $\{R^{\kappa}_{\alpha,\beta}(t)\}_{t>0}$ then $\{R^{\kappa}_{\alpha,\beta}(t)\}_{t>0}$ is a (b, g_{β})-regularized family as well (according to [27]), and we have the following result, see [27]. See also [1, Definition 2.3 and Remark 2.4] and [2, Section 4]

Proposition 2.4. *If $\alpha, \beta > 0$ and A generates an (α,β)-resolvent family $\{R^{\kappa}_{\alpha,\beta}(t)\}_{t>0}$, then*

(1) $\lim_{t\to 0^+} \frac{R^{\kappa}_{\alpha,\beta}(t)x}{g_{\beta}(t)} = x$, *for all $x \in X$,*
(2) *$R^{\kappa}_{\alpha,\beta}(t)x \in D(A)$ and $R^{\kappa}_{\alpha,\beta}(t)Ax = AR^{\kappa}_{\alpha,\beta}(t)x$ for all $x \in D(A)$ and $t > 0$*
(3) *For all $x \in D(A)$,*

$$R^{\kappa}_{\alpha,\beta}(t)x = g_{\beta}(t)x + \int_0^t b(t-s) A R^{\kappa}_{\alpha,\beta}(s) x ds,$$

(4) *$\int_0^t b(t-s) R^{\kappa}_{\alpha,\beta}(s) x ds \in D(A)$ and*

$$R^{\kappa}_{\alpha,\beta}(t)x = g_{\beta}(t)x + A \int_0^t b(t-s) R^{\kappa}_{\alpha,\beta}(s) x ds,$$

for all $x \in X$,

where $b(t) = g_{\alpha}(t) + (\kappa * g_{\alpha})(t)$.

The next result gives a subordination theorem for (α,β)-resolvent families.

Theorem 2.5 (Subordination)**.** *Let $0 < \alpha < 1$ and $\varepsilon \geq 0$. Let A be the generator of a resolvent family $\{R^a(t)\}_{t\geq 0}$ and $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$ be a given kernel. Suppose that $\|R^a(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, where $M, \omega \geq 0$. Assume that there exist $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\nu \leq 0$ and such that $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda) > \nu$. Then, A is the generator of the $(\alpha, \alpha+\varepsilon)$-resolvent family $\{R^\kappa_{\alpha,\alpha+\varepsilon}(t)\}_{t>0}$ defined by*

$$R^\kappa_{\alpha,\alpha+\varepsilon}(t)x := \int_0^\infty \psi_{\alpha,\varepsilon}(t,s)R^a(s)xds, \quad t > 0, x \in X \tag{2.10}$$

where $\psi_{\alpha,\varepsilon}$ is the Wright type function given in Definition 2.2. Moreover, if $\varepsilon > 0$, then

$$R^\kappa_{\alpha,\alpha+\varepsilon}(t)x = (g_\varepsilon * R^\kappa_{\alpha,\alpha})(t)x, \tag{2.11}$$

for all $x \in X$ and $t > 0$.

Proof. The exponential boundedness and strong continuity of $\{R^\kappa_{\alpha,\alpha+\varepsilon}(t)\}_{t>0}$ follows as in the proof of [2, Theorem 4.5]. See also [29, Chapter I, Section 4]. We give here only the details in order to prove that $\{R^\kappa_{\alpha,\alpha+\varepsilon}(t)\}_{t>0}$ given by (2.10) defines an $(\alpha, \alpha+\varepsilon)$-resolvent family. In fact, since $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda) > \nu$ and $\{R^a(t)\}_{t\geq 0}$ is an exponentially bounded resolvent family, by (2.8) we can apply the Fubini's theorem, and by the same equation (2.8) we obtain

$$\begin{aligned}
\int_0^\infty e^{-\lambda t}R^\kappa_{\alpha,\alpha+\varepsilon}(t)xdt &= \int_0^\infty e^{-\lambda t}\left(\int_0^\infty \psi_{\alpha,\varepsilon}(t,s)R^a(s)xds\right)dt \\
&= \int_0^\infty \left(\int_0^\infty e^{-\lambda t}\psi_{\alpha,\varepsilon}(t,s)dt\right)R^a(s)xds \\
&= \lambda^{-\varepsilon}\int_0^\infty e^{-\lambda^\alpha s}R^a(s)xds \\
&= \frac{\lambda^{-\varepsilon}}{1+\hat{a}(\lambda^\alpha)}\left(\frac{\lambda^\alpha}{1+\hat{a}(\lambda^\alpha)} - A\right)^{-1}x \\
&= \frac{\lambda^{\alpha-(\alpha+\varepsilon)}}{1+\hat{\kappa}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)} - A\right)^{-1}x
\end{aligned}$$

for all λ such that $\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)} \in \rho(A)$. The identity (2.11) follows directly from the properties of the Laplace transform, concluding the proof. ☐

Corollary 2.6. *Let $0 < \alpha < 1$. Let A be the generator of a resolvent family $\{R^a(t)\}_{t\geq 0}$ and $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$ be a given kernel. Suppose that $\|R^a(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, where $M, \omega \geq 0$. Assume that there exist $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\nu \leq 0$ such that $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda) > \nu$. Then, A is the generator of the resolvent families $\{R^\kappa_{\alpha,\alpha}(t)\}_{t>0}$ and $\{R^\kappa_{\alpha,1}(t)\}_{t>0}$ which are, respectively, defined by*

$$R^\kappa_{\alpha,\alpha}(t)x := \int_0^\infty \psi_{\alpha,0}(t,s)R^a(s)xds, \quad t > 0, \tag{2.12}$$

and

$$R^\kappa_{\alpha,1}(t)x := \int_0^\infty \psi_{\alpha,1-\alpha}(t,s)R^a(s)xds, \quad t > 0. \tag{2.13}$$

Moreover, if f is Laplace transformable, then the unique mild solution to Problem (1.4) *is given by*

$$u(t) \;=\; R^\kappa_{\alpha,1}(t)x + \int_0^t R^\kappa_{\alpha,\alpha}(t-s)f(s)ds. \tag{2.14}$$

Proof. For $0 < \alpha < 1$, we observe that (2.12) and (2.13) follow with $\varepsilon = 0$ and $\varepsilon = 1-\alpha$, respectively, in the Subordination Theorem 2.5.

On the other hand, the Laplace transform of $R^\kappa_{\alpha,\alpha}(t)$ and $R^\kappa_{\alpha,1}(t)$ are respectively given by

$$\widehat{R^\kappa_{\alpha,\alpha}}(\lambda)=\frac{1}{1+\hat{\kappa}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)}-A\right)^{-1} \quad \text{and} \quad \widehat{R^\kappa_{\alpha,1}}(\lambda)=\frac{\lambda^{\alpha-1}}{1+\hat{\kappa}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)}-A\right)^{-1},$$

for all λ such that $\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)}\in\rho(A)$. Taking Laplace transform in both sides of (1.4) we obtain by (2.7) that

$$\hat{u}(\lambda)=\frac{\lambda^{\alpha-1}}{1+\hat{\kappa}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)}-A\right)^{-1}x+\frac{1}{1+\hat{\kappa}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)}-A\right)^{-1}\hat{f}(\lambda)$$

which implies that u is given by (2.14), by the uniqueness of the Laplace transform. □

We notice that by [26, Formula (3.8) in Corollay 3.3] and [26, Corollay 3.3 (b)], the Wright type functions $\psi_{\alpha,0}$ and $\psi_{\alpha,1-\alpha}$ in Corollary 2.6 can be written, respectively as

$$\psi_{\alpha,0}(t,s)=\frac{1}{\pi}\int_0^\infty e^{t\rho\cos\theta-s\rho^\alpha\cos\alpha\theta}\cdot\sin(t\rho\sin\theta-s\rho\sin\alpha\theta+\theta)d\rho,$$

for $\pi/2<\theta<\pi$ and

$$\psi_{\alpha,1-\alpha}(t,s)=\frac{1}{\pi}\int_0^\infty \rho^{\alpha-1}e^{-s\rho^\alpha\cos\alpha(\pi-\theta)-t\rho\cos\theta}\cdot\sin\left(t\rho\sin\theta-s\rho^\alpha\sin\alpha(\pi-\theta)+\alpha(\pi-\theta)\right)d\rho, \tag{2.15}$$

for $\theta\in(\pi-\frac{\pi}{2\alpha},\pi/2)$. Moreover, since

$$\widehat{R^\kappa_{\alpha,1}}(\lambda)=\frac{1}{\lambda^{1-\alpha}}\frac{1}{1+\hat{\kappa}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{\kappa}(\lambda)}-A\right)^{-1}=\hat{g}_{1-\alpha}(\lambda)\widehat{R^\kappa_{\alpha,\alpha}}(\lambda)$$

we obtain, by the uniqueness of the Laplace transform, that $R^\kappa_{\alpha,1}(t)$ can be written as

$$R^\kappa_{\alpha,1}(t)=(g_{1-\alpha}*R^\kappa_{\alpha,\alpha})(t),\quad t>0.$$

On the other hand, we notice that in the Hypotheses of Theorem 2.5, given a kernel $\kappa\in L^1_{\text{loc}}(\mathbb{R}_+)$ we need to find a kernel $a\in L^1_{\text{loc}}(\mathbb{R}_+)$ such that $\hat{a}(\lambda^\alpha)=\hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda)>\nu$. Now, we show some examples of such kernels.

Example 2.7. If $\gamma\in\mathbb{R}$, $\rho,\mu>0$, and $\kappa(t)=\gamma e^{-\rho t}\frac{t^{\mu-1}}{\Gamma(\mu)}$, then $\hat{\kappa}(\lambda)=\frac{\gamma}{(\lambda+\rho)^\mu}$, for all $\lambda>-\rho$, which means that the kernel a needs to verify $\hat{a}(\lambda^\alpha)=\gamma/(\lambda+\rho)^\mu$, that is $\hat{a}(\lambda)=\frac{\gamma}{(\lambda^{\frac{1}{\alpha}}+\rho)^\mu}$. By [16, Formula (11.13), p.13] *we conclude that*

$$a(t)=\gamma t^{\frac{\mu}{\alpha}-1}E^\mu_{\frac{1}{\alpha},\frac{\mu}{\alpha}}(-\rho t^{\frac{1}{\alpha}}), \tag{2.16}$$

where, for $p,q,r>0$, $E^r_{p,q}(z)$ is the generalized Mittag-Leffler type function defined by

$$E^r_{p,q}(z)=\sum_{j=0}^\infty\frac{(r)_j z^j}{j!\Gamma(pj+q)},\quad z\in\mathbb{C},$$

where $(r)_j$ denotes the Pochhammer symbol defined by $(r)_j=\frac{\Gamma(r+j)}{\Gamma(r)}$. The kernel κ (known as the relaxation kernel) appears in problems of viscoelasticity or heat conduction with memory, see for instance [9] *and* [29, Chapter I, Section 5]*. Fractional differential equations in the form of* (1.4) *with this kernel have been recently studied in* [3, 8, 30]*. In particular, if $\mu=1$, then $\kappa(t)=\gamma e^{-\rho t}$, and therefore*

$$a(t)=\gamma t^{\frac{1}{\alpha}-1}E_{\frac{1}{\alpha},\frac{1}{\alpha}}(-\rho t^{\frac{1}{\alpha}}).$$

This exponential kernel $\kappa(t)$ typically appears when one consider Maxwell materials in viscoelasticity theory. In that context, the parameters γ and ρ are given by $\gamma=\tau$ and $\rho=\tau/\varpi$, where τ corresponds to the elastic modulus of the material and ϖ to the coefficient of viscosity, see for instance [10] *and* [29, Chapter II, Section 9].

Remark 2.8. *We remark that, very recently, the authors in* [1] *have studied the existence and uniqueness of local and global mild solutions, blow up and critical nonlinearities for*

$$\left\{\begin{array}{l}\partial_t^\alpha u(t)=Au(t)+\displaystyle\int_0^t \kappa(t-s)Au(s)ds+f(t,u(t)),\ t\in[0,T]\\ u(0)=x,\end{array}\right. \tag{2.17}$$

where $T>0$, $-A$ *is a sectorial operator defined in a Banach space* X, $x\in X$, f *is a continuous function and* $\kappa(t)=-e^{-\rho t}\frac{t^{\mu-1}}{\Gamma(\mu)}$, *where* $\rho\ge 0$ *and* $0<\mu\le 1$. *More concretely, in* [1, Theorem 1.1] *the authors proved that* A *generates the resolvent families* $\{S(t)\}_{t>0}$ *and* $\{R(t)\}_{t>0}$ *given by*

$$S(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{\lambda^{\alpha-1}(\lambda+\rho)^\mu}{(\lambda+\rho)^\mu-1}\left(\frac{\lambda^\alpha(\lambda+\rho)^\mu}{(\lambda+\rho)^\mu-1}-A\right)^{-1}d\lambda,\quad t>0,$$

and

$$R(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\frac{(\lambda+\rho)^\mu}{(\lambda+\rho)^\mu-1}\left(\frac{\lambda^\alpha(\lambda+\rho)^\mu}{(\lambda+\rho)^\mu-1}-A\right)^{-1}d\lambda,\quad t>0,$$

(where Γ *is a suitable Hankel's path) which correspond, respectively, to the resolvent families* $\{R^\kappa_{\alpha,\alpha}(t)\}_{t>0}$ *and* $\{R^\kappa_{\alpha,1}(t)\}_{t>0}$ *given in Corollary 2.6.*

On the other hand, in order to have the conclusions in Theorem 2.5 we need to find conditions on the operator A and on the kernel κ implying that A is the generator of a resolvent family $\{R^a(t)\}_{t\ge 0}$. Now, we study such conditions.

A linear operator $A:D(A)\subset X\to X$ is said to be ω-*sectorial of angle* θ if there are constants $\omega\in\mathbb{R}$, $M>0$ and $\theta\in(\pi/2,\pi)$ such that $\rho(A)\supset S_{\theta,\omega}:=\{z\in\mathbb{C}:z\neq\omega:|\arg(z-\omega)|<\theta\}$ and

$$\|(z-A)^{-1}\|\le\frac{M}{|z-\omega|}\quad\text{for all}\quad z\in S_{\theta,\omega}. \tag{2.18}$$

In order to simplify the presentation of the results, we assume that $\omega=0$. In that case, we write $A\in\mathrm{Sect}(\theta,M)$ and we denote the sector $S_{\theta,0}$ as S_θ. These operators have been studied widely, both in abstract settings (see for instance [4, 15]) and for their applications in the study of linear and nonlinear integro/differential equations, see for example [11, 19, 28, 34].

From (1.2) we now consider the homogeneous problem

$$\left\{\begin{array}{l}u'(t)=Au(t)+\displaystyle\int_0^t a(t-s)Au(s)ds,\ t\ge 0\\ u(0)=x,\end{array}\right. \tag{2.19}$$

where A with domain $D(A)$ is a densely defined linear sectorial operator on the Banach space X, $x\in X$ and the kernel $a\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ satisfies $\hat a(\lambda^\alpha)=\hat\kappa(\lambda)$ for all $\mathrm{Re}(\lambda)>\nu$. It is easy to verify that (2.19) is equivalent to the integral equation

$$u(t)=x+\int_0^t c(t-s)Au(s)ds,\quad t\ge 0, \tag{2.20}$$

where $c(t):=1+(1*a)(t)$, and by [29, Chapter I], the well-posedness of (2.19) (or equivalently, the integral equation (2.20)) is equivalent to the existence of the resolvent $\{R^a(t)\}_{t\ge 0}$. In this case, for each $x\in X$, the unique mild solution to (2.19) is given by

$$u(t)=R^a(t)x,\quad t\ge 0. \tag{2.21}$$

Definition 2.9. [29, p. 69] *Let* $k\in\mathbb{N}$. *Let* $b\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ *be of subexponential growth, which means*

$$\int_0^\infty e^{-\varepsilon t}|b(t)|dt<\infty,\qquad \textit{for all } \varepsilon>0.$$

The kernel $b(t)$ is called 1-regular *(of constant c) if there is a constant $c > 0$ such that*

$$|\lambda \hat{b}'(\lambda)| \leq c|\hat{b}(\lambda)|, \quad \textit{for all} \quad \operatorname{Re}(\lambda) > 0, \tag{2.22}$$

where $\hat{b}'(\lambda)$ is the derivative of $\hat{b}(\lambda)$ with respect to λ.

Remark 2.10.

We notice that if b is a 1-regular kernel of constant $c \leq 1$ with $\hat{b}(\lambda) \neq -1$ for all $\lambda \in \mathbb{C}$, then $\frac{|\hat{b}(\lambda)|}{|1+\hat{b}(\lambda)|} \leq 1$ for all $\operatorname{Re}(\lambda) > 0$. In fact, let $\lambda = re^{i\phi}$ with $0 \leq \phi \leq \frac{\pi}{2}$ and $r > 0$. For $g(\lambda) := \hat{b}(\lambda)$ we have

$$\arg\left(\hat{b}(\lambda)\right) = \operatorname{Im}(\ln(g(re^{i\phi}))) = \operatorname{Im}\int_0^{\phi} \frac{d}{dt}\ln(g(re^{it}))dt = \operatorname{Im}\int_0^{\phi} \frac{g'(re^{it})ire^{it}}{g(re^{it})}dt.$$

Since b is a 1-regular kernel, there exists a constant $c > 0$ such that $|\lambda \hat{b}'(\lambda)| \leq c|\hat{b}(\lambda)|$, for all $\operatorname{Re}(\lambda) > 0$, and we obtain

$$|\arg\left(\hat{b}(\lambda)\right)| \leq \operatorname{Im}\int_0^{\phi} \frac{|g'(re^{it})ire^{it}|}{|g(re^{it})|}dt \leq c\phi \leq \frac{\pi}{2},$$

which implies that $\operatorname{Re}(\hat{b}(\lambda)) > 0$ for all $\operatorname{Re}(\lambda) > 0$ and therefore $\frac{|\hat{b}(\lambda)|}{|1+\hat{b}(\lambda)|} \leq 1$, for all $\operatorname{Re}(\lambda) > 0$.

Theorem 2.11 (Generation)**.** *Let $A \in \operatorname{Sect}(\theta, M)$ be a sectorial operator. Suppose that b is a 1-regular kernel with $\hat{b}(\lambda) \neq -1$ for all $\operatorname{Re}(\lambda) > \nu$, where $\nu \leq 0$. If the constant c in* (2.22) *satisfies $(1+c)\frac{\pi}{2} \leq \theta$, then A is the generator of a resolvent family $\{R^b(t)\}_{t\geq 0}$.*

Proof. Define $h(\lambda) = \frac{\lambda}{1+\hat{b}(\lambda)}$. We write $\lambda = re^{i\phi}$ with $|\phi| \leq \frac{\pi}{2}$ and $r > 0$. We notice that we may assume that $\phi \geq 0$. Then

$$\arg\left(\frac{\lambda}{1+\hat{b}(\lambda)}\right) = \operatorname{Im}(\ln(h(re^{i\phi}))) = \operatorname{Im}\int_0^{\phi} \frac{d}{dt}\ln(h(re^{it}))dt = \operatorname{Im}\int_0^{\phi} \frac{h'(re^{it})ire^{it}}{h(re^{it})}dt.$$

An easy computation shows that

$$\frac{\lambda h'(\lambda)}{h(\lambda)} = 1 - \frac{\lambda \hat{b}'(\lambda)}{1+\hat{b}(\lambda)}$$

and therefore

$$\arg\left(\frac{\lambda}{1+\hat{b}(\lambda)}\right) = \operatorname{Im}\int_0^{\phi}\left(1 - \frac{re^{it}\hat{b}'(re^{it})}{1+\hat{b}(re^{it})}\right)dt.$$

The 1-regularity of b, the hypothesis and Remark 2.10 imply that

$$\left|\arg\left(\frac{\lambda}{1+\hat{b}(\lambda)}\right)\right| \leq \operatorname{Im}\int_0^{\phi}\left(1 + \frac{c|\hat{b}(re^{it})|}{|1+\hat{b}(re^{it})|}\right)dt \leq (1+c)\phi \leq (1+c)\frac{\pi}{2} \leq \theta.$$

This inequality shows that $h(\lambda) \in S_\theta$ and thus the function H given by $H(\lambda) := \frac{1}{1+\hat{b}(\lambda)}\left(\frac{\lambda}{1+\hat{b}(\lambda)} - A\right)^{-1}$ is well-defined. Since A is a sectorial operator, we obtain

$$\|\lambda H(\lambda)\| \leq |h(\lambda)|\|(h(\lambda) - A)^{-1}\| \leq M.$$

On the other hand, since

$$\lambda^2 H'(\lambda) = -\frac{\lambda \hat{b}'(\lambda)}{1+\hat{b}(\lambda)}\lambda H(\lambda) - (\lambda H(\lambda))^2\left(1 - \frac{\lambda \hat{b}'(\lambda)}{1+\hat{b}(\lambda)}\right),$$

the 1-regularity of b implies that

$$\|\lambda^2 H'(\lambda)\| \leq \frac{c|\hat{b}(\lambda)|}{|1+\hat{b}(\lambda)|}M + M^2\left(1 + \frac{c|\hat{b}(\lambda)|}{|1+\hat{b}(\lambda)|}\right) \leq cM + (1+c)M^2.$$

Therefore,

$$\|\lambda H(\lambda)\| + \|\lambda^2 H'(\lambda)\| \leq (1+c)(M+M^2) =: M_1,$$

for all $\mathrm{Re}(\lambda) > 0$ and by [29, Proposition 0.1] we conclude

$$\|H^{(n)}(\lambda)\| \leq \frac{M_1 n!}{\lambda^{n+1}}, \quad n \in \mathbb{N}_0, \lambda > 0,$$

and thus, for every $\mu > 0$ we obtain

$$\|H^{(n)}(\lambda)\| \leq \frac{M_1 n!}{(\lambda-\mu)^{n+1}}, \quad n \in \mathbb{N}_0, \lambda > \mu.$$

Finally, from the generation Theorem [29, Theorem 1.3] we conclude that A is the generator of a resolvent family $\{R^b(t)\}_{t\geq 0}$ such that $\|R^b(t)\| \leq M_1 e^{\mu t}$ for all $t \geq 0$. □

Lemma 2.12. *Let $\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be a given kernel. Suppose that $\frac{1}{2} < \alpha < 1$ and there exist $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\nu \leq 0$ such that $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\mathrm{Re}(\lambda) > \nu$. Then κ is a 1-regular kernel if and only if a is a 1-regular kernel.*

Proof. Suppose that κ is a 1-regular kernel. If a is not a 1-regular kernel, there exists $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) > 0$ such that

$$|\lambda_0 \hat{a}'(\lambda_0)| \geq n|\hat{a}(\lambda_0)|,$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$ we obtain $\hat{a}(\lambda_0) = 0$. For this λ_0 we have two options: $\mathrm{Re}(\lambda_0^{\frac{1}{\alpha}}) < 0$ or $\mathrm{Re}(\lambda_0^{\frac{1}{\alpha}}) > 0$. If $\mathrm{Re}(\lambda_0^{\frac{1}{\alpha}}) < 0$, then $\mathrm{Re}(-\lambda_0^{\frac{1}{\alpha}}) > 0$ and therefore $\mathrm{Re}((-\lambda_0^{\frac{1}{\alpha}})^\alpha) > 0$ (because $\alpha < 1$). If we write $\lambda_0 = re^{i\phi}$ with $0 < \phi < \frac{\pi}{2}$ then we obtain $(-\lambda_0^{\frac{1}{\alpha}})^\alpha = re^{(\phi+\alpha\pi)i}$. Since $0 < \phi < \frac{\pi}{2}$, $\frac{1}{2} < \alpha < 1$ we obtain $\frac{\pi}{2} < \phi + \alpha\pi < \frac{3\pi}{2}$, which is impossible because $\mathrm{Re}((-\lambda_0^{\frac{1}{\alpha}})^\alpha) > 0$. Therefore, $\mathrm{Re}(\lambda_0^{\frac{1}{\alpha}}) > 0 \geq \nu$. We notice that $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\mathrm{Re}(\lambda) > \nu$ if and only if $\hat{a}(\lambda) = \hat{\kappa}(\lambda^{\frac{1}{\alpha}})$ for all $\mathrm{Re}(\lambda^{\frac{1}{\alpha}}) > \nu$ and therefore $0 = \hat{a}(\lambda_0) = \hat{\kappa}(\lambda_0^{\frac{1}{\alpha}})$, which means that $\hat{\kappa}$ has a zero in the open right halfplane, a contradiction, because κ is a 1-regular kernel (see [29, Section 3.2, p.69]).

Conversely, if a is a 1-regular kernel, then there exists a positive constant c such that $|\lambda\hat{a}'(\lambda)| \leq c|\hat{a}(\lambda)|$ for all $\mathrm{Re}(\lambda) > 0$. Since $\frac{1}{2} < \alpha < 1$ we conclude that if $\mathrm{Re}(\lambda) > 0$, then $\mathrm{Re}(\lambda^\alpha) > 0 \geq \nu$ and $|\lambda\hat{\kappa}'(\lambda)| = |\lambda^\alpha \hat{a}'(\lambda^\alpha)| \leq c|\hat{a}(\lambda^\alpha)| = c|\hat{\kappa}(\lambda)|$. □

Remark 2.13. *An easy computation shows that the kernels κ given in Example 2.7 are 1-regular and therefore (by Lemma 2.12) the corresponding kernels $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ are 1-regular as well ($\frac{1}{2} < \alpha < 1$) and satisfy the hypotheses in Generation Theorem 2.11.*

We notice that if $\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is a 1-regular kernel, then there exists $c > 0$ such that $|\lambda\hat{\kappa}'(\lambda)| \leq c|\hat{\kappa}(\lambda)|$ for all $\mathrm{Re}(\lambda) > 0 \geq \nu$. Moreover, if $\frac{1}{2} < \alpha < 1$, then the kernel a satisfying $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\mathrm{Re}(\lambda) > \nu$ is 1-regular as well, and therefore there exists $d > 0$ such that $|\lambda\hat{a}'(\lambda)| \leq d|\hat{a}(\lambda)|$ for all $\mathrm{Re}(\lambda) > 0 \geq \nu$. Thus $|\lambda\hat{a}'(\lambda)| = \frac{1}{\alpha}|\lambda^{\frac{1}{\alpha}}\hat{k}'(\lambda^{\frac{1}{\alpha}})| \leq \frac{c}{\alpha}|\hat{k}(\lambda^{\frac{1}{\alpha}})| = \frac{c}{\alpha}|\hat{a}(\lambda)|$, which implies that $d \leq \frac{c}{\alpha}$. Since $\hat{\kappa}(\lambda) = \hat{a}(\lambda^\alpha)$, we obtain, similarly, that $|\lambda\hat{\kappa}'(\lambda)| = \alpha|\lambda^\alpha\hat{a}'(\lambda^\alpha)| \leq \alpha d|\hat{a}(\lambda^\alpha)| = \alpha d|\hat{\kappa}(\lambda)|$ and therefore $c \leq \alpha d$. We conclude that κ is a 1-regular kernel of constant $c > 0$ if and only if a is a 1-regular of constant $\frac{c}{\alpha}$.

On the other hand, for the given kernel $\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, in the Subordination Theorem 2.5 we need to assume the existence of $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\nu \leq 0$ such that $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\mathrm{Re}(\lambda) > \nu$ and that A is the generator of a resolvent family $\{R^a(t)\}_{t\geq 0}$. The existence of such kernel a is closely related to completely positive and Bernstein functions (see for instance [29, Chapter I, Section 4]). The Generation Theorem 2.11 shows that if $A \in \mathrm{Sect}(\theta, M)$ and the constant c verifies the condition $(1 + \frac{c}{\alpha})\frac{\pi}{2} \leq \theta$, then A generates a resolvent family $\{R^a(t)\}_{t\geq 0}$, and therefore, the Lemma 2.7 and Corollary 2.6 imply that if $\frac{1}{2} < \alpha < 1$ and $\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is a 1-regular kernel, then A is the generator of the resolvent families $\{R^\kappa_{\alpha,\alpha}(t)\}_{t>0}$ and $\{R^\kappa_{\alpha,1}(t)\}_{t>0}$ defined, respectively, by (2.12) and (2.13).

We conclude that in order to study the existence of mild solutions to Problem (1.4) for a given 1-regular kernel $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$ we only need to find the kernel $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\nu \leq 0$ satisfying $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda) > \nu$. The above conclusions are summarized in the next result.

Theorem 2.14. *Let $A \in \text{Sect}(\theta, M)$ be a sectorial operator, $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\frac{1}{2} < \alpha < 1$. Assume that there exist $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\nu \leq 0$ such that $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda) > \nu$. Suppose that κ is a 1-regular kernel and that the constant c in (2.22) satisfies $(1 + \frac{c}{\alpha})\frac{\pi}{2} \leq \theta$. If f is Laplace transformable, then the Problem (1.4) has a unique mild solution u given by*

$$u(t) \quad = \quad R^\kappa_{\alpha,1}(t)x + \int_0^t R^\kappa_{\alpha,\alpha}(t-s)f(s)ds,$$

where $\{R^\kappa_{\alpha,1}(t)\}_{t>0}$ and $\{R^\kappa_{\alpha,\alpha}(t)\}_{t>0}$ are given in Corollary 2.6.

3. Examples

In this section, we consider some examples of the operator A and we obtain the explicit solution to (1.4).

3.1. **The case where $A = \varrho I$ and $\kappa(t) = \gamma \frac{t^{\mu-1}}{\Gamma(\mu)} e^{-\rho t}$.** Suppose that $A = \varrho I$ for some $\varrho > 0$, and assume that $\rho, \mu > 0$, $\gamma \in \mathbb{R} \setminus \{0\}$ and $\frac{1}{2} < \alpha < 1$. The homogeneous initial value problem (1.4) reads

$$\begin{cases} \partial_t^\alpha u(t) = \varrho u(t) + \dfrac{\varrho\gamma}{\Gamma(\mu)} \displaystyle\int_0^t e^{-\rho(t-s)}(t-s)^{\mu-1}u(s)ds, \quad t > 0, \\ u(0) = x. \end{cases} \tag{3.23}$$

If $\text{Re}(\lambda) > 0$, then

$$\frac{|\lambda\hat{\kappa}'(\lambda)|}{|\hat{\kappa}(\lambda)|} = \frac{\mu|\lambda|}{|\lambda + \rho|} \leq \mu$$

which means that κ is a 1-regular kernel with constant $c = \mu$. The Example 2.7 shows that there exist a kernel $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $\nu := -\rho \leq 0$ such that $\hat{a}(\lambda^\alpha) = \hat{\kappa}(\lambda)$ for all $\text{Re}(\lambda) > \nu$, and therefore, the Lemma 2.12 implies that a is a 1-regular kernel as well (with constant $\frac{\mu}{\alpha}$). On the other hand, we notice that A is a sectorial operator of angle θ for all $\theta \in (\pi/2, \pi)$ and thus for $0 < \mu < \frac{1}{2}$ we have $(1 + \mu/\alpha)\frac{\pi}{2} \leq \theta$. By the Generation Theorem 2.11, the operator A generates a resolvent family $\{R^a(t)\}_{t\geq 0}$ and by Subordination Theorem 2.5 A is also a generator of an $(\alpha, 1)$-resolvent family $\{R^\kappa_{\alpha,1}(t)\}_{t>0}$. Therefore, the solution u to (3.23) is given by $u(t) = R^\kappa_{\alpha,1}(t)x$ for all $t > 0$. We notice that in this case, $R^\kappa_{\alpha,1}(t)$ can be found explicitly. In fact, since $\hat{a}(\lambda) = \gamma/(\lambda^{\frac{1}{\alpha}} + \rho)^\mu$ and A generates the resolvent family $\{R^a(t)\}_{t\geq 0}$, then

$$\widehat{R^a}(\lambda) = \frac{1}{1 + \hat{a}(\lambda)} \left(\frac{\lambda}{1 + \hat{a}(\lambda)} - A \right)^{-1} = \frac{(\lambda^{\frac{1}{\alpha}} + \rho)^\mu}{\lambda(\lambda^{\frac{1}{\alpha}} + \rho)^\mu - \varrho(\lambda^{\frac{1}{\alpha}} + \rho)^\mu - \gamma\varrho} = \frac{1}{(\lambda - \varrho)} \frac{1}{1 - \frac{\gamma\varrho}{(\lambda-\varrho)(\lambda^{\frac{1}{\alpha}}+\rho)^\mu}}.$$

Since

$$\left| \frac{\gamma\varrho}{(\lambda - \varrho)(\lambda^{\frac{1}{\alpha}} + \rho)^\mu} \right| < 1,$$

for λ large enough, we obtain that

$$\widehat{R^a}(\lambda) = \sum_{k=0}^\infty \frac{\varrho^k}{(\lambda - \varrho)^{k+1}} \left(\frac{\gamma}{(\lambda^{\frac{1}{\alpha}} + \rho)^\mu} \right)^k,$$

and by [16, Formula (11.13), p.13] we conclude that

$$R^a(t) = \sum_{k=0}^\infty \frac{\varrho^k\gamma^k}{k!} \int_0^t (t-s)^k e^{\varrho(t-s)} s^{\frac{\mu k}{\alpha}-1} E^{\mu k}_{\frac{1}{\alpha},\frac{\mu k}{\alpha}}(-\rho s^{\frac{1}{\alpha}})ds$$

and therefore, the solution u to (3.23) is given by (see Corollary 2.6)

$$u(t)=R^{\kappa}_{\alpha,1}(t)x=\sum_{k=0}^{\infty}\frac{\varrho^k\gamma^k}{k!}\int_0^{\infty}\psi_{\alpha,1-\alpha}(t,s)\int_0^s(s-r)^k e^{\varrho(s-r)}r^{\frac{\mu k}{\alpha}-1}E^{\mu k}_{\frac{1}{\alpha},\frac{\mu k}{\alpha}}(-\rho r^{\frac{1}{\alpha}})xdrds, \tag{3.24}$$

where $\psi_{\alpha,1-\alpha}$ is given in (2.15). We have proved the following result.

Proposition 3.15. *Suppose that $\varrho>0$, $\gamma\in\mathbb{R}$ and $\frac{1}{2}<\alpha<1$. If $0<\mu<\frac{1}{2}$, then the unique solution u to the Problem (3.23) is given by (3.24).*

3.2. **The case of a self-adjoint operator and $\kappa(t)=\gamma\frac{t^{\mu-1}}{\Gamma(\mu)}e^{-\rho t}$.** Now, suppose that $-A$ is a non-negative and self-adjoint operator on the Hilbert space $L^2(\Omega)$ where $\Omega\subset\mathbb{R}^N$ is a bounded open set. If the operator A has a compact resolvent, then $-A$ has a discrete spectrum and its eigenvalues satisfy $0\le\lambda_1\le\lambda_2\le\cdots\le\lambda_n\le\cdots$ and $\lim_{n\to\infty}\lambda_n=\infty$.

Denote by ϕ_n to the the normalized eigenfunction associated with λ_n. It is well known that $\{\phi_n : n\in\mathbb{N}\}$ is an orthonormal basis for $L^2(\Omega)$, it is also total in $D(A)$ and for all $v\in D(A)$ we have

$$-Av=\sum_{k=1}^{\infty}\lambda_n\langle v,\phi_n\rangle_{L^2(\Omega)}\phi_n.$$

As a consequence of Proposition 3.15, we have the following result.

Corollary 3.16. *Let A be an operator as above. Suppose that $\kappa(t)=\gamma\frac{t^{\mu-1}}{\Gamma(\mu)}e^{-\rho t}$, where $\gamma\in\mathbb{R}\setminus\{0\}$. If $0<\mu<\frac{1}{2}$ and $\frac{1}{2}<\alpha<1$, then the unique solution u to the Problem*

$$\begin{cases}\partial_t^{\alpha}u(t,x)=Au(t,x)+\displaystyle\int_0^t\kappa(t-s)Au(s,x)ds,\ t\ge0,\\ u(0,x)=u_0(x),\end{cases} \tag{3.25}$$

where $x\in\Omega$ and $u_0\in L^2(\Omega)$ is given by

$$u(t,x)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{\varrho^k\lambda_n^k}{k!}\int_0^{\infty}\psi_{\alpha,1-\alpha}(t,s)\int_0^s(s-r)^k e^{\varrho(s-r)}r^{\frac{\mu k}{\alpha}-1}E^{\mu k}_{\frac{1}{\alpha},\frac{\mu k}{\alpha}}(-\rho r^{\frac{1}{\alpha}})u_0(x)drds,\quad t\ge0, \tag{3.26}$$

where $\psi_{\alpha,1-\alpha}$ is given in (2.15).

Proof. Let A be as above. Consider the problem

$$\begin{cases}u'(t,x)=Au(t,x)+\displaystyle\int_0^t a(t-s)Au(s,x)ds,\ t\ge0,x\in\Omega\\ u(0)=u_0(x),\end{cases} \tag{3.27}$$

where a is given by (2.16). Multiplying both sides of (3.27) by $\phi_n(x)$ and integrating over Ω we obtain that $u_n(t):=\langle u(t),\phi_n\rangle_{L^2(\Omega)}$ is a solution of the system

$$\begin{cases}u_n'(t,x)=-\lambda_n u_n(t,x)-\lambda_n\displaystyle\int_0^t a(t-s)u_n(s,x)ds,\ t>0,x\in\Omega,\\ u_n(0,x)=u_{0,n}(x),\end{cases} \tag{3.28}$$

where $u_{0,n}=\langle u_0,\phi_n\rangle_{L^2(\Omega)}$, for all $n\in\mathbb{N}$. Then, the solution to (3.28) is given by $u_n(t,x)=R^a(t)u_0(x)$, $t\ge0,x\in\Omega$, where $R^a(t)$ is given by

$$R^a(t)=\sum_{k=0}^{\infty}\frac{\varrho^k\lambda_n^k}{k!}\int_0^{\infty}\psi_{\alpha,1-\alpha}(t,s)\int_0^s(s-r)^k e^{\varrho(s-r)}r^{\frac{\mu k}{\alpha}-1}E^{\mu k}_{\frac{1}{\alpha},\frac{\mu k}{\alpha}}(-\rho r^{\frac{1}{\alpha}})drds.$$

Since κ is a 1-regular kernel with $\hat{a}(\lambda^{\alpha})=\hat{\kappa}(\lambda)$ for all $\mathrm{Re}(\lambda)>-\rho$ and A is a sectorial operator of angle θ for all $\theta\in(\pi/2,\pi)$ and $0<\mu<\frac{1}{2}$ we conclude by Theorem 2.14 and Proposition 3.15 that the solution to (3.25) is given by (3.26) and the proof is finished. □

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