Hölder continuous solutions for fractional differential equations and maximal regularity

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Abstract

We characterize the existence and uniqueness of solutions of an abstract fractional differential equation with infinite delay in Hölder spaces.

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1. Introduction

In this paper, we consider the following fractional differential equation with infinite delay

$$D^{\beta}u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where A is a closed linear operator defined on a Banach space X, $a \in L^1(\mathbb{R}_+)$ is a scalar-valued kernel, $f \in C^{\alpha}(\mathbb{R}; X)$, $0 < \alpha < 1$, and the fractional derivative for $\beta > 0$ is taken in the sense of Caputo.

Fractional differential equations have been used by many researchers to adequately describe the evolution of a variety of physical and biological processes. Examples include studies in electrochemistry, electromagnetism, viscoelasticity, rheology, among other. See, for instance [1, 26, 31] and [34] for further details.

When $\beta = 1$ in equation (1.1), we obtain the equation with infinite delay

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \qquad t \in \mathbb{R}. \tag{1.2}$$

Equations of this kind arise, for example, in the study of heat flow in materials with memory as well as in some equations of population dynamics or in viscoelasticity. In such applications the operator A is typically the Laplacian,

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the elasticity operator, the Stokes operator, or the biharmonic Δ^2, among other. See [38] and [40, Chapter III, Section 13] for further details.

In [4], Arendt, Batty and Bu study the existence and uniqueness of Hölder continuous solutions to equation (1.1) in the case $\beta = 1$ and $a \equiv 0$, that is,

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}. \tag{1.3}$$

These authors introduce the notion of C^α-*multipliers* and prove an operator-valued Fourier multiplier theorem for Hölder spaces on the line. As a consequence, the authors obtain a characterization, by means of a simple resolvent estimates of the underlying operator, of the *well-posedness* of (1.3), in the sense that there exists a unique classical solution to equation (1.3) in Hölder spaces.

Using the results of Arendt, Batty and Bu ([4]) about vector-valued Fourier multipliers, some characterizations of the existence and uniqueness of solutions (on the real line) for several classes of equations in Hölder spaces, have been obtained in the last years. See [12, 18, 30, 39] for further information.

Existence of Hölder continuous solutions to fractional differential equations in the form of (1.1) have been studied for example, by Clement, Gripenberg and Londen using the method of the sum of Da Prato and Grisvard [16]. See moreover El-Sayed and Herzallah [21, 22, 23] and references therein. Other approaches to the existence and uniqueness of solutions to fractional differential equations can be found, for example, in [6, 9, 13, 17, 20, 32]. The obtained results give sufficient conditions to the existence and uniqueness of Hölder continuous solutions to equations in the form of (1.1), but leaves as an open problem to *characterize* the existence and uniqueness of Hölder solutions to fractional differential equations.

Characterizations of the existence and uniqueness of solutions to the linear problem (1.1) have been studied only on periodic vector-valued Lebesgue spaces, $L^p_{2\pi}(\mathbb{R}; X)$, $1 < p < \infty$, (where X is a UMD space) and in the scale of periodic Besov spaces $B^s_{p,q}([0, 2\pi]; X)$ (and therefore on periodic Hölder space $C^s([0, 2\pi]; X)$) by Bu [10, 11]. The main tool in these results are two operator-valued Fourier multipliers theorems of Arendt and Bu [5, 7] on periodic vector-valued spaces $L^p_{2\pi}(\mathbb{R}; X)$, $1 < p < \infty$, and $B^s_{p,q}([0, 2\pi]; X)$. Using these results on operator-valued multipliers, other fractional differential equations on periodic vector-valued spaces have been recently studied in [29, 33]. In all these results is obtained a relation between the existence and uniqueness of solutions to fractional differential equations and the $\mathcal{R}$-boundedness of a sequence of operators (see [19, 27, 41]).

In this paper, we apply the method of operator-valued Fourier multipliers on the line (see [4]) to characterize the well-posedness (or maximal regularity) of the fractional differential equation (1.1) in $C^\alpha(\mathbb{R}; X)$, the vector-valued Hölder spaces for $0 < \alpha < 1$. More specifically, we show in Theorem 3.7 that if a is a 2-regular kernel, then the problem (1.1) is C^α-well posed if and only if

$$\frac{(i\eta)^\beta}{1 + \tilde{a}(\eta)} \in \rho(A), \quad \text{for all } \eta \in \mathbb{R} \quad \text{and} \quad \sup_{\eta \in \mathbb{R}} \left\| \frac{(i\eta)^\beta}{1 + \tilde{a}(\eta)} \left(\frac{(i\eta)^\beta}{1 + \tilde{a}(\eta)} - A \right)^{-1} \right\| < \infty,$$

where $\tilde{a}$ denotes the Fourier transform of a, (more precisely of their extension to $\mathbb{R}$ by setting them equal to 0 on $(-\infty, 0)$).

We notice that, here the operator A is not necessarily the generator of a C_0-semigroup and there is not any restriction on the Banach space X.

It is remarkable that in the scalar case, that is $A = \rho I$, where $\rho \in \mathbb{R} \setminus \{0\}$, and $a \equiv 0$, we have that the unique Hölder solution to fractional differential equation

$$D^\beta u(t) = \rho u(t) + f(t), \qquad t \in \mathbb{R},$$

is explicitly given by

$$u(t) = \int_{-\infty}^{t} (t-s)^{\beta-1} E_{\beta,\beta}(\rho(t-s)^\beta) f(s) ds, \quad t \in \mathbb{R},$$

where $E_{\beta,\beta}$ denotes the Mittag-Leffler function. See Example 5.13 below.

The paper is organized as follows. In Section 2, we review some results about vector-valued Fourier multipliers and we recall the definition and some basic properties on fractional calculus. Section 3 is devoted to our main result (Theorem 3.7), where a characterization of the well-posedness (or maximal regularity) of equation (1.1) is obtained under some suitable assumptions on kernel a. In Section 4, the equation $D^\beta u(t) = A^\gamma u(t) + f(t)$, where $\gamma > 0$, is studied as a particular case of equation (1.1). Finally, some examples are examined in Section 5.

2. Preliminaries

Let X and Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ be the space of all bounded linear operators from X to Y. If $X = Y$, we write simply $\mathcal{B}(X)$. Let $0 < \alpha < 1$. We denote by $C^\alpha(\mathbb{R}; X)$ the space of all X-valued functions f on $\mathbb{R}$, such that

$$\|f\|_\alpha := \sup_{t \neq s} \frac{\|f(t) - f(s)\|}{|t-s|^\alpha} < \infty.$$

If we define $\|f\|_{C^\alpha} := \|f\|_\alpha + \|f(0)\|$, then $C^\alpha(\mathbb{R}; X)$ is a Banach space under the norm $\|\cdot\|_{C^\alpha}$.

The kernel of the seminorm $\|\cdot\|_\alpha$ on $C^\alpha(\mathbb{R}; X)$ is the space of all constant functions and the corresponding quotient space $\dot{C}^\alpha(\mathbb{R}; X)$ is a Banach space in the induced norm. We identify a function $f \in C^\alpha(\mathbb{R}; X)$ with its equivalence class

$$\dot{f} := \{g \in C^\alpha(\mathbb{R}; X) : f - g \equiv \text{ constant}\}.$$

In this way, $\dot{C}^\alpha(\mathbb{R}; X)$ may be identified with the space of all $f \in C^\alpha(\mathbb{R}; X)$ such that $f(0) = 0$. See [4, Section 5].

For $n \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, $C^n(\mathbb{R}; X)$ denotes the set of X-valued functions which are n-times differentiable on $\mathbb{R}$.

Given $\beta > 0$, the *Liouville fractional integrals of order* β, $D_-^{-\beta} f$ and $D_+^{-\beta} f$ are defined, respectively, by

$$D_-^{-\beta} f(t) := \int_{-\infty}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)ds, \quad t \in \mathbb{R}, \tag{2.1}$$

and

$$D_+^{-\beta} f(t) := \int_{t}^{\infty} \frac{(s-t)^{\beta-1}}{\Gamma(\beta)} f(s)ds, \quad t \in \mathbb{R}. \tag{2.2}$$

A sufficient condition for that the fractional integrals (2.1) and (2.2) exist is that $f(t) = O(|t|^{-\beta-\epsilon})$ for $\epsilon > 0$ and $t \to \infty$. Integrable functions satisfying this property are sometimes referred to as functions of Liouville class, see [36].

The *Caputo left and right-sided fractional derivatives,* corresponding to those in (2.1) and (2.2) are defined, respectively, by

$$D_-^{\beta} f(t) := D_-^{-(n-\beta)} \frac{d^n}{dt^n} f(t) = \int_{-\infty}^{t} \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)} f^{(n)}(s)ds \tag{2.3}$$

and

$$D_+^{\beta} f(t) := (-1)^n D_+^{-(n-\beta)} \frac{d^n}{dt^n} f(t) = (-1)^n \int_{t}^{\infty} \frac{(s-t)^{n-\beta-1}}{\Gamma(n-\beta)} f^{(n)}(s)ds, \tag{2.4}$$

where $t \in \mathbb{R}$, $f \in C^n(\mathbb{R}; X)$ and $n = \lceil \beta \rceil$. Here $\lceil \beta \rceil$ denotes the the smallest integer greater than or equal to β. More details of Caputo fractional calculus can be found in [31, Section 2.4] and [14].

We notice that the Caputo fractional calculus can also be applied to functions not belonging to the Liouville class (see [36, p. 237]). For example, if g and h are measurable functions on $\mathbb{R}$ such that $D_\pm^{-\beta} g$ exists and $h = D_\pm^{-\beta} g$ a.e., then we set $D_\pm^{\beta} h = g$.

It is known that $D_\pm^{\beta+\gamma} = D_\pm^{\beta}(D_\pm^{\gamma})$ for any $\beta, \gamma \in \mathbb{R}$, where $D_\pm^0 = \mathrm{Id}$ denotes the identity operator and $(-1)^n D_+^n = D_-^n = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$. See [36].

In what follows, we refer to the Caputo left-sided fractional derivative, $D_-^{\beta} f$, as the *Caputo fractional derivative of order* $\beta > 0$ *of* f and we write $D^{\beta} f := D_-^{\beta} f$. For example, for the function $e^{\lambda t}$ we have

$$D_-^{-\beta} e^{\lambda t} = \lambda^{-\beta} e^{\lambda t} \quad \text{and} \quad D^{\beta} e^{\lambda t} = \lambda^{\beta} e^{\lambda t}, \quad \mathrm{Re}\lambda \geq 0.$$

For $\beta > 0$, let $C^{\alpha,\beta}(\mathbb{R}, X)$ be the Banach space of all $u \in C^n(\mathbb{R}, X)$, $n = \lceil \beta \rceil$, such that $D^{\beta} u$ exists and belongs to $C^{\alpha}(\mathbb{R}, X)$ equipped with the norm

$$\|u\|_{C^{\alpha,\beta}} = \|D^{\beta} u\|_{C^{\alpha}} + \sum_{j=1}^{n} \|D^{\beta-j} u(0)\|.$$

We denote by $\mathcal{F} f$ the Fourier transform of f, that is

$$(\mathcal{F} f)(s) := \tilde{f}(s) := \int_{\mathbb{R}} e^{-ist} f(t)dt,$$

for $s \in \mathbb{R}$ and $f \in L^1(\mathbb{R}; X)$.

Let $\Omega \subset \mathbb{R}$ be an open set. By $C_c^\infty(\Omega)$ we denote the space of all $C^\infty-$functions in Ω having compact support in Ω.

Definition 2.1. *Let $M : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X, Y)$ be continuous. We say that M is a $\dot{C}^\alpha-$*multiplier *if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R}; X) \to \dot{C}^\alpha(\mathbb{R}; Y)$ such that*

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot M))(s)f(s)ds \tag{2.5}$$

for all $f \in C^\alpha(\mathbb{R}; X)$ and all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Here $(\mathcal{F}(\phi \cdot M))(s) = \int_{\mathbb{R}} e^{-ist}\phi(t)M(t)dt \in \mathcal{B}(X, Y)$. Observe that the right-hand side of (2.5) does not depend on the representative of $\dot{f}$ since

$$\int_{\mathbb{R}} (\mathcal{F}(\phi M)(s))(s)ds = 2\pi(\phi M)(0) = 0.$$

Therefore, if L exists, then it is well defined. Moreover, left-hand side of (2.5) determines the function $Lf \in C^\alpha(\mathbb{R}; X)$ uniquely up to some constant (by [4, Lemma 5.1]). Moreover, if (2.5) holds, then $L : \dot{C}^\alpha(\mathbb{R}; X) \to \dot{C}^\alpha(\mathbb{R}; Y)$ is linear and continuous (see [4, Definition 5.2]) and if $f \in C^\alpha(\mathbb{R}; X)$ is bounded, then Lf is bounded as well (see [4, Remark 6.3]).

The following multiplier theorem is due to Arendt, Batty and Bu [4, Theorem 5.3].

Theorem 2.2. *Let $M \in C^2(\mathbb{R} \setminus \{0\}, \mathcal{B}(X, Y))$ be such that*

$$\sup_{t \neq 0} \|M(t)\| + \sup_{t \neq 0} \|tM'(t)\| + \sup_{t \neq 0} \|t^2 M''(t)\| < \infty. \tag{2.6}$$

Then, M is a $\dot{C}^\alpha-$multiplier.

Remark 2.3.

Recall that a Banach space X has the *Fourier type* p, with $1 \leq p \leq 2$, if the Fourier transform defines a bounded linear operator from $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; X)$, where $1/p + 1/q = 1$. As examples, $L^p(\Omega)$, with $1 \leq p \leq 2$, has Fourier type p; the Banach space X has the Fourier type 2 if and only if X is isomorphic to a Hilbert space; X has Fourier type p if and only if X^* has Fourier type p. Every Banach space has Fourier type 1. A Banach space X is say to be $B-$convex if it has Fourier type p, for some $p > 1$. Every uniformly convex space is $B-$convex.

If X is $B-$convex, in particular if X is a UMD space, then the Theorem 2.2 holds if the condition (2.6) is replaced by the weaker condition

$$\sup_{t \neq 0} \|M(t)\| + \sup_{t \neq 0} \|tM'(t)\| < \infty, \tag{2.7}$$

where $M \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{B}(X, Y))$, see [4, Remark 5.5].

Let $0 < \alpha < 1$. We denote by $L^1(\mathbb{R}_+, t^\alpha dt)$ the set of all $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ such that

$$\int_0^\infty |a(t)|t^\alpha dt < \infty. \tag{2.8}$$

Observe that as consequence such a is always in $L^1(\mathbb{R}_+)$.

Given $v \in C^\alpha(\mathbb{R}; X)$ $(0 < \alpha < 1)$ and $a \in L^1(\mathbb{R}_+, t^\alpha dt)$, we write

$$(a\dot{*}v)(t) := \int_{-\infty}^t a(t-s)v(s)ds = \int_0^\infty a(s)v(t-s)ds. \tag{2.9}$$

From (2.8) the above integral is well defined. Moreover, it follows from the definition that

$$\text{if } v \in C^\alpha(\mathbb{R}; X) \text{ then } a\dot{*}v \in C^\alpha(\mathbb{R}; X) \text{ and } \|a\dot{*}v\|_\alpha \leq \|a\|_1 \|v\|_\alpha. \tag{2.10}$$

The Laplace transform of a function $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$ is denoted by

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \text{Re}\lambda > \omega,$$

whenever the integral is absolutely convergent for $\text{Re}\lambda > \omega$. The relation between the Laplace transform of $f \in L^1(\mathbb{R}; X)$, $f(t) = 0$ for $t < 0$, and its Fourier transform is

$$\mathcal{F}(f)(s) = \hat{f}(is), \quad s \in \mathbb{R}.$$

For $f \in L^1_{\text{loc}}(\mathbb{R}; X)$ of subexponential growth, that is

$$\int_{-\infty}^\infty e^{-\epsilon|t|} \|f(t)\| dt < \infty, \qquad \text{for each } \epsilon > 0,$$

we denote by $\hat{f}(\lambda)$ for the *Carleman transform of f* :

$$\hat{f}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} f(t)dt, & \text{Re}\lambda > 0, \\ \\ -\int_{-\infty}^0 e^{-\lambda t} f(t)dt, & \text{Re}\lambda < 0. \end{cases}$$

Observe that we use the same symbol for the Carleman and Laplace transform but, this will not lead to confusion.

When $f \in L^1(\mathbb{R}; X)$ is of subexponential growth, we have by [8, Chapter 4],

$$\lim_{\sigma \to 0^+} (\hat{f}(\sigma + i\rho) - \hat{f}(-\sigma + i\rho)) = \tilde{f}(\rho), \quad \rho \in \mathbb{R}. \tag{2.11}$$

If $a \in L^1(\mathbb{R}_+)$, we will always identify a with its extension on $\mathbb{R}$ by letting $a(t) = 0$ for $t < 0$. In such way, when $a \in L^1(\mathbb{R}_+)$, the Fourier transform $\tilde{a}(\rho)$ makes sense for all $\rho \in \mathbb{R}$. Moreover, by (2.11) we have

$$\lim_{\sigma \to 0^+} \hat{a}(\sigma + i\rho) = \tilde{a}(\rho)$$

and $\hat{a}(-\sigma + i\rho) = 0$ for all $\sigma > 0$ and $\rho \in \mathbb{R}$ by definition.

In what follows, we always assume that $\tilde{a}(\eta) \neq -1$, for all $\eta \in \mathbb{R}$, and we use the following notation:

$$a_\eta := \tilde{a}(\eta), \quad \eta \in \mathbb{R}.$$

Now, we recall the notion of regular kernels (see [40, p. 69]).

Definition 2.4. *Let $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. $a(t)$ is called k-*regular *if there is a constant $c > 0$ such that*

$$|\lambda^n [\hat{a}(\lambda)]^{(n)}| \leq c|\hat{a}(\lambda)|, \quad \textit{for all} \quad \operatorname{Re}(\lambda) > 0, 0 \leq n \leq k.$$

For the reader's convenience, we summarize here from [40, Lemma 8.1] some properties of 1-regular kernels.

Lemma 2.5. *Suppose that $b \in L^1_{\text{loc}}(\mathbb{R}_+)$ is of sub-exponential growth and* 1-*regular. Then*

(i) $\hat{b}(i\rho) := \lim_{\lambda \to i\rho} \hat{b}(\lambda)$ exists for each $\rho \neq 0$;

(ii) $\hat{b}(\lambda) \neq 0$ for each $Re(\lambda) \geq 0$, $\lambda \neq 0$;

(iii) $\hat{b}(i\cdot) \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \setminus \{0\})$;

(iv) $|\rho[\hat{b}(i\rho)]'| \leq c|\hat{b}(i\rho)|$ for a.a. $\rho \in \mathbb{R}$.

By $L^1(\mathbb{R}, (1+|t|)^{-k}dt; X)$ we denote the space of all functions $f \in L^1_{\text{loc}}(\mathbb{R}; X)$ such that $\int_{\mathbb{R}} \|f(t)\|(1 + |t|)^{-k}dt < \infty$ for some $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Note that $L^1(\mathbb{R}; X) \subseteq L^1(\mathbb{R}, (1 + |t|)^{-k}dt; X)$ for all $k \in \mathbb{N}_0$. See [8, Section 4.8].

Observe that if $f \in C^\alpha(\mathbb{R}; X)$ then $\|f(t)\| \leq c(1+|t|^\alpha)$ for a suitable constant c. Therefore, $f \in L^1(\mathbb{R}, (1 + |t|)^{-k}dt; X)$ for some $k \in \mathbb{N}_0$.

Let $f \in L^1(\mathbb{R}, (1 + |t|)^{-k}dt; X)$, where $k \in \mathbb{N}_0$. We define $\mathcal{F}f$ as a linear mapping from $C_c^\infty(\mathbb{R} \setminus \{0\})$ into X by

$$\langle \varphi, \mathcal{F}f \rangle = \int_{\mathbb{R}} f(t)(\mathcal{F}\varphi)(t)dt, \quad \varphi \in C_c^\infty(\mathbb{R} \setminus \{0\}).$$

The next lemma follows from [8, Theorems 4.8.1 and 4.8.2].

Lemma 2.6. *Let $f \in L^1(\mathbb{R}, (1+|t|)^{-k}dt; X)$, where $k \in \mathbb{N}_0$. Then f is constant if and only if $\langle \varphi, \mathcal{F}f \rangle = \int_{\mathbb{R}} f(s)(\mathcal{F}\varphi)(s)ds = 0$ for all $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.*

3. A characterization

In this section, we characterize the C^α-well posedness of the following fractional differential equation

$$D^\beta u(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)ds + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $A : D(A) \subseteq X \to X$ is a linear and closed operator, $a \in L^1(\mathbb{R}_+, t^\alpha dt)$, $\beta > 0$, and $f \in C^\alpha(\mathbb{R}, X)$, $0 < \alpha < 1$. As in [4] we define the map $\text{id} : \mathbb{R} \to \mathbb{C}$

by $\mathrm{id}(s) = is$. The function id^β is defined by $\mathrm{id}^\beta(s) = (is)^\beta$, where $(is)^\beta = |s|^\beta e^{\frac{\pi\beta i}{2}\mathrm{sgn}(s)}$ (here $\mathrm{sgn}(s)$ denotes the sign of s).

The Caputo left and right-sided fractional derivatives are adjoint in the sense of the following lemma.

Lemma 3.1. *If $D^\beta f$ and $D_+^{-\beta} g$ exist, then*

$$\int_{\mathbb{R}} f(t)g(t)dt = \int_{\mathbb{R}} D^\beta f(t) D_+^{-\beta} g(t)dt.$$

Proof. By Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}} g(t)D^{-\beta}f(t)dt &= \int_{\mathbb{R}} g(t)\int_{-\infty}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)dsdt \\
&= \int_{\mathbb{R}} f(s)\int_{s}^{\infty} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(t)dtds \\
&= \int_{\mathbb{R}} f(s)D_+^{-\beta}g(s)ds.
\end{aligned}$$

Therefore, we obtain

$$\int_{\mathbb{R}} f(t)g(t)dt = \int_{\mathbb{R}} D^\beta f(t) D_+^{-\beta} g(t)dt.$$

□

The following Lemma is a generalization of [4, Lemma 6.2].

Lemma 3.2. *Let $0 < \alpha < 1$, $u, v \in C^\alpha(\mathbb{R}; X)$ and $\beta > 0$. Then, the following assertions are equivalent,*

(i) $u \in C^{\alpha,\beta}(\mathbb{R}; X)$ and $D^\beta u - v$ is constant;

(ii) $\int_{\mathbb{R}} v(s)\mathcal{F}(\phi)(s)ds = \int_{\mathbb{R}} u(s)\mathcal{F}(\mathrm{id}^\beta \cdot \phi)(s)ds$, for all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Proof. $(i) \Rightarrow (ii)$. Let $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Observe that using Fubini's Theorem, we have for $t \in \mathbb{R}$,

$$\begin{aligned}
D_+^{-\beta}\mathcal{F}(\mathrm{id}^\beta \cdot \phi)(t) &= \int_t^\infty \frac{(s-t)^{\beta-1}}{\Gamma(\beta)} \mathcal{F}(\mathrm{id}^\beta \cdot \phi)(s)ds \\
&= \int_t^\infty \int_{\mathbb{R}} \frac{(s-t)^{\beta-1}}{\Gamma(\beta)} e^{-isr}(ir)^\beta \phi(r)drds \\
&= \int_{\mathbb{R}} \int_0^\infty \frac{v^{\beta-1}}{\Gamma(\beta)} e^{-i(t+v)r}(ir)^\beta \phi(r)dvdr \\
&= \int_{\mathbb{R}} \int_0^\infty \frac{v^{\beta-1}}{\Gamma(\beta)} e^{-ivr}[e^{-itr}(ir)^\beta \phi(r)]dvdr \\
&= (\mathcal{F}\phi)(t).
\end{aligned}$$

Hence, for $t \in \mathbb{R}$ we have $D_+^{-\beta}\mathcal{F}(\mathrm{id}^\beta \cdot \phi)(t) = (\mathcal{F}\phi)(t)$ and thus $\mathcal{F}(\mathrm{id}^\beta \cdot \phi)(t) = D_+^\beta \mathcal{F}(\phi)(t)$. Therefore, by hypothesis and Lemmas 2.6, 3.1, we obtain

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} D^\beta u(s) D_+^{-\beta}\mathcal{F}(\mathrm{id}^\beta \cdot \phi)(s)ds = \int_{\mathbb{R}} u(s)\mathcal{F}(\mathrm{id}^\beta \cdot \phi)(s)ds.$$

(ii) $\Rightarrow$ *(i)*. Let $\phi \in C_c^\infty(\mathbb{R}\backslash\{0\})$ and $\psi(s) = \phi(s)/(is)^\beta$. An easy computation shows that $\mathcal{F}(\phi)(s) = \mathcal{F}(\mathrm{id}^\beta \cdot \psi)(s) = D_+^\beta(\mathcal{F}\psi)(s)$.

Since $u \in C^\alpha(\mathbb{R}; X)$ there exists $k \in \mathbb{N}_0$ such that $u \in L^1(\mathbb{R}, (1+|t|)^{-k}dt; X)$. Take $k_0 \in \mathbb{N}$ with $k_0 > k + \beta$. We claim that $D^\beta u \in L^1(\mathbb{R}, (1+|t|)^{-k_0}dt; X)$. In fact, by Fubini's theorem, by [36, p. 249] and Lemma 3.1 we have

$$\begin{aligned} \int_{\mathbb{R}} D^\beta u(t)(1+|t|)^{-k_0}dt &= \frac{\Gamma(k_0+\beta)}{\Gamma(k_0)} \int_{\mathbb{R}} D^\beta u(t) D_+^{-\beta}(1+|t|)^{-(\beta+k_0)}dt \\ &= \frac{\Gamma(k_0+\beta)}{\Gamma(k_0)} \int_{\mathbb{R}} u(t)(1+|t|)^{-(\beta+k_0)}dt. \end{aligned}$$

Since $u \in L^1(\mathbb{R}, (1+|t|)^{-k}dt; X)$ we conclude that

$$\begin{aligned} \int_{\mathbb{R}} \|D^\beta u(t)\|(1+|t|)^{-k_0}dt &\leq \frac{\Gamma(k_0+\beta)}{\Gamma(k_0)} \int_{\mathbb{R}} \|u(t)\|(1+|t|)^{-(\beta+k_0)}dt \\ &\leq \frac{\Gamma(k_0+\beta)}{\Gamma(k_0)} \int_{\mathbb{R}} \|u(t)\|(1+|t|)^{-k}ds < \infty. \end{aligned}$$

On the other hand, by the Lemma 3.1 we obtain

$$\begin{aligned} \langle \phi, \mathcal{F}v \rangle = \int_{\mathbb{R}} v(s)(\mathcal{F}\phi)(s)ds &= \int_{\mathbb{R}} u(s)\mathcal{F}(\mathrm{id}^\beta \cdot \phi)(s)ds \\ &= \int_{\mathbb{R}} u(s) D_+^\beta(\mathcal{F}\phi)(s)ds \\ &= \int_{\mathbb{R}} D^\beta u(s)(\mathcal{F}\phi)(s)ds = \langle \phi, \mathcal{F}D^\beta u \rangle. \end{aligned}$$

We conclude that $\langle \phi, \mathcal{F}(v - D^\beta u) \rangle = 0$ for all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$, and therefore $v - D^\beta u$ is constant by Lemma 2.6. $\square$

The following Lemma, is a direct consequence of [30, Lemma 3.2].

Lemma 3.3. *Let $0 < \alpha < 1$, $v \in C^\alpha(\mathbb{R}; [D(A)])$, $u \in C^\alpha(\mathbb{R}; X)$ and $a \in L^1(\mathbb{R}_+, t^\alpha dt)$. The following assertions are equivalent,*

(i) $a * Av - u$ is constant;

(ii) $\int_{\mathbb{R}} u(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} Av(s)\mathcal{F}(a_s\phi)(s)ds$, for all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Definition 3.4. *We say that the equation (3.1) is C^α-*well posed *(or has* maximal regularity*) if, for each $f \in C^\alpha(\mathbb{R}; X)$, there exists a unique function $u \in C^\alpha(\mathbb{R}; [D(A)]) \cap C^{\alpha,\beta}(\mathbb{R}; X)$, such that the equation (3.1) holds for all $t \in \mathbb{R}$.*

Remark 3.5.

Observe that if (3.1) is C^α-well posed, it follows from the closed graph theorem that the map $L : C^\alpha(\mathbb{R}; X) \to C^\alpha(\mathbb{R}; [D(A)]) \cap C^{\alpha,\beta}(\mathbb{R}; X)$, which associates to f the unique solution u of (3.1) is linear and continuous. Indeed, since A is a closed operator, we have that the space $H := C^\alpha(\mathbb{R}; [D(A)]) \cap C^{\alpha,\beta}(\mathbb{R}; X)$ endowed with the norm

$$\|u\|_H := \|D^\beta u\|_{C^\alpha} + \|Au\|_{C^\alpha} + \|a \dot{*} Au\|_{C^\alpha} + \|u\|_{C^\alpha}$$

is a Banach space.

Proposition 3.6. *Let $A : D(A) \subseteq X \to X$ be a closed linear operator defined on a Banach space X and $\beta > 0$. Suppose that the problem (3.1) is C^α-well posed. Then,*

(i) $\dfrac{(i\eta)^\beta}{1 + a_\eta} \in \rho(A)$ *for all $\eta \in \mathbb{R}$, and*

(ii) $\displaystyle\sup_{\eta \in \mathbb{R}} \left\| \frac{(i\eta)^\beta}{1 + a_\eta} \left(\frac{(i\eta)^\beta}{1 + a_\eta} - A \right)^{-1} \right\| < \infty.$

Proof. Let $\eta \in \mathbb{R}$ and suppose that

$$((i\eta)^\beta - (1 + a_\eta)A)x = 0 \tag{3.2}$$

where $x \in D(A)$. Let $u(t) = e^{i\eta t}x$. Then, u is a solution to (3.1) with $f \equiv 0$. In fact, $D^\beta u(t) = (i\eta)^\beta e^{i\eta t}x$ (see [36, p. 248]). Moreover, by (3.2) we have

$$Au(t) + (a \dot{*} Au)(t) = e^{i\eta t}(1 + a_\eta)Ax = e^{i\eta t}(i\eta)^\beta x = D^\beta u(t).$$

Hence, by uniqueness it follows that $u \equiv 0$, that is, $x = 0$. We obtain that $((i\eta)^\beta - (1 + a_\eta)A) = (1 + a_\eta)\left(\frac{(i\eta)^\beta}{1+a_\eta} - A\right)$ is injective. Therefore, $\left(\frac{(i\eta)^\beta}{1+a_\eta} - A\right)$ is injective.

In order to show the surjectivity, let $y \in X$. Consider the bounded operator $L : C^\alpha(\mathbb{R}; X) \to C^\alpha(\mathbb{R}; [D(A)]) \cap C^{\alpha,\beta}(\mathbb{R}; X)$ which takes each $f \in C^\alpha(\mathbb{R}; X)$ to the unique solution u of equation (3.1). Let $\eta \in \mathbb{R}$, $f(t) = e^{i\eta t}y$ and $u = Lf$. Then, for fixed $s \in \mathbb{R}$ we have that $v_1(t) := u(t+s)$ and $v_2(t) := e^{i\eta s}u(t)$ are both solutions of (3.1) with $g(t) = e^{is\eta}f(t)$. Hence, $v_1 = v_2$, that is, $u(t+s) = e^{is\eta}u(t)$ for all $s, t \in \mathbb{R}$. Let $x = u(0) \in D(A)$. Then, $u(t) = e^{i\eta t}x$ (i.e. $u(t) = u(t + 0) = v_1(t)$) satisfies the equation (3.1). Now, observe that

$$(a \dot{*} Au)(t) = e^{i\eta t}a_\eta Ax, \quad t \in \mathbb{R}.$$

In particular, $(a \dot{*} Au)(0) = a_\eta Ax$. Since $D^\beta u(t) = (i\eta)^\beta e^{i\eta t}x$ we have $D^\beta u(0) = (i\eta)^\beta x$ and therefore,

$$((i\eta)^\beta - (1 + a_\eta)A)x = D^\beta u(0) - Au(0) - a_\eta Au(0).$$

Since $u(t)$ satisfies the equation (3.1) for all $t \in \mathbb{R}$, we obtain,

$$((i\eta)^\beta - (1 + a_\eta)A)x = D^\beta u(0) - Au(0) - a_\eta Au(0) = f(0) = y. \tag{3.3}$$

Therefore $((i\eta)^\beta - (1+a_\eta)A) = (1+a_\eta)\left(\frac{(i\eta)^\beta}{1+a_\eta} - A\right)$ is surjective. Thus, we conclude that $\left(\frac{(i\eta)^\beta}{1+a_\eta} - A\right)$ is surjective. Since A is a closed operator, we have $\frac{(i\eta)^\beta}{1+a_\eta} \in \rho(A)$ for all $\eta \in \mathbb{R}$.

By (3.3) we have $u(t) = e^{i\eta t}((i\eta)^\beta - (1+a_\eta)A)^{-1}y$. Denote $e_\eta \otimes x$ the function $t \mapsto (e_\eta \otimes x)(t) := e^{i\eta t}x$. Since $\|e_\eta \otimes x\|_\alpha = \gamma_\alpha |\eta|^\alpha \|x\|$, where $\gamma_\alpha = 2\sup_{t>0} t^{-\alpha}\sin(t/2)$ (see [4, Section 3]) we have

$$\begin{aligned}
\gamma_\alpha|\eta|^\alpha\|(i\eta)^\beta((i\eta)^\beta-(1+a_\eta)A)^{-1}y\| &= \|e_\eta\otimes(i\eta)^\beta((i\eta)^\beta-(1+a_\eta)A)^{-1}y\|_\alpha \\
&= \|D^\beta u\|_\alpha \le \|D^\beta u\|_{C^\alpha} \le \|u\|_{C^{\alpha,\beta}} \\
&= \|Lf\|_{C^{\alpha,\beta}} \le \|L\|\,\|f\|_{C^\alpha} \\
&\le \|L\|(\|f\|_\alpha + \|f(0)\|) \\
&= \|L\|(\gamma_\alpha|\eta|^\alpha + 1)\|y\|.
\end{aligned}$$

Therefore, $\|(i\eta)^\beta((i\eta)^\beta-(1+a_\eta)A)^{-1}\| \le \|L\|(1+\gamma_\alpha^{-1}|\eta|^{-\alpha})$ and in consequence $\sup_{|\eta|\le 1}\|(i\eta)^\beta((i\eta)^\beta-(1+a_\eta)A)^{-1}\| < \infty$. By continuity it follows that

$$\sup_{\eta\in\mathbb{R}}\left\|\frac{(i\eta)^\beta}{1+a_\eta}\left(\frac{(i\eta)^\beta}{1+a_\eta} - A\right)^{-1}\right\| = \sup_{\eta\in\mathbb{R}}\|(i\eta)^\beta((i\eta)^\beta-(1+a_\eta)A)^{-1}\| < \infty.$$

□

The following Theorem is the main result in this paper, which shows that under an additional hypothesis (the 2-regularity of kernel a) we can prove the converse of Proposition 3.6.

Theorem 3.7. *Let $A : D(A) \subseteq X \to X$ be a linear closed operator defined on Banach space X and $a \in L^1(\mathbb{R}_+, t^\alpha dt)$. Suppose that the kernel a is 2-regular and satisfies $\sup_{\eta\in\mathbb{R}}|\frac{1}{1+a_\eta}| < \infty$. Then, the following assertions are equivalent*

(i) The equation (3.1) is C^α-well posed;

(ii) $\frac{(i\eta)^\beta}{1+a_\eta} \in \rho(A)$ for all $\eta\in\mathbb{R}$ and $\sup_{\eta\in\mathbb{R}}\left\|\frac{(i\eta)^\beta}{1+a_\eta}\left(\frac{(i\eta)^\beta}{1+a_\eta} - A\right)^{-1}\right\| < \infty$.

Proof. $(ii) \Rightarrow (i)$. For $t\in\mathbb{R}$, define the operator $N(t) := ((it)^\beta - (1+a_t)A)^{-1}$. Observe that by hypothesis $N \in C^2(\mathbb{R}; \mathcal{B}(X,[D(A)]))$. We claim that N is a $\dot{C}^\alpha$-multiplier. In fact, the identity $(it)^\beta N(t) = (1+a_t)AN(t) + I$ and the hypothesis imply that $\sup_{t\in\mathbb{R}}\|N(t)\| < \infty$. On the other hand,

$$N'(t) = -N(t)[i\beta(it)^{\beta-1} - a_t'A]N(t), \quad \text{and}$$

$$\begin{aligned}
N''(t) &= 2N(t)[i\beta(it)^{\beta-1} - a_t'A]N(t)[i\beta(it)^{\beta-1} - a_t'A]N(t) \\
&\quad -N(t)[i^2\beta(\beta-1)(it)^{\beta-2} - a_t''A]N(t).
\end{aligned}$$

Hence,

$$\|tN'(t)\| \le \beta\|(it)^\beta N(t)\| + \|a_tAN(t)\|,$$

$$\begin{aligned}\|t^2N''(t)\| \quad &\le \quad 2\beta^2\|N(t)\|\cdot\|(it)^\beta N(t)\|^2 + 2\beta\|N(t)(it)^\beta N(t)a_tAN(t)\| + \\ &\quad\ 2\beta\|N(t)a_tAN(t)(it)^\beta N(t)\| + 2\|N(t)\|\cdot\|a_tN(t)\|^2 + \\ &\quad\ \beta|\beta-1|\cdot\|N(t)(it)^\beta N(t)\| + \|N(t)a_tAN(t)\|.\end{aligned}$$

From the identity $(it)^\beta N(t) = (1+a_t)AN(t)+I$ we have that $\sup_{t\in\mathbb{R}}\|a_tAN(t)\| < \infty$ and from the 2-regularity of a we obtain that

$$\sup_{t\in\mathbb{R}}\|tN'(t)\| < \infty \text{ and } \sup_{t\in\mathbb{R}}\|t^2N''(t)\| < \infty.$$

We conclude from Theorem 2.2 that the operator N is a $\dot{C}^\alpha$-multiplier, with $N \in C^2(\mathbb{R};\mathcal{B}(X,[D(A)]))$.

Define the operator $S \in C^2(\mathbb{R};\mathcal{B}(X))$ by $S(t) := (\mathrm{id}^\beta\cdot N)(t)$. Observe that by hypothesis $\sup_{t\in\mathbb{R}}\|S(t)\| < \infty$. On the other hand,

$$S'(t) = i\beta(it)^{\beta-1}N(t) + (it)^\beta N'(t),$$

$$S''(t) = \beta(\beta-1)i^2(it)^{\beta-2}N(t) + 2i\beta(it)^{\beta-1}N'(t) + (it)^\beta N''(t),$$

and

$$tS'(t) = \beta(it)^\beta N(t) + (it)^\beta tN'(t),$$
$$t^2S''(t) = \beta(\beta-1)(it)^\beta N(t) + 2\beta(it)^\beta tN'(t) + (it)^\beta t^2N''(t).$$

Hence, from hypothesis we conclude $\sup_{t\in\mathbb{R}}\|tS'(t)\| < \infty$ and $\sup_{t\in\mathbb{R}}\|t^2S''(t)\| < \infty$. Thus, S is a $\dot{C}^\alpha$-multiplier by Theorem 2.2. A similar computation shows that $T \in C^2(\mathbb{R};\mathcal{B}(X))$ defined by $T(t) := a_tAN(t)$ is a $\dot{C}^\alpha$-multiplier.

Let $f \in C^\alpha(\mathbb{R};X)$. Since N, S and T are $\dot{C}^\alpha$-multipliers, there exist $\overline{u} \in C^\alpha(\mathbb{R};[D(A)])$, $v \in C^\alpha(\mathbb{R};X)$ and $w \in C^\alpha(\mathbb{R};X)$ such that

$$\int_{\mathbb{R}} \overline{u}(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi\cdot N)(s)f(s)ds, \tag{3.4}$$

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\varphi\cdot S)(s)f(s)ds, \tag{3.5}$$

$$\int_{\mathbb{R}} w(s)(\mathcal{F}\psi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\psi\cdot T)(s)f(s)ds, \tag{3.6}$$

for all $\phi,\varphi,\psi \in C_c^\infty(\mathbb{R}\setminus\{0\})$. Letting $\phi = \mathrm{id}^\beta\cdot\varphi$ in (3.4) we obtain from (3.5)

$$\int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(\mathrm{id}^\beta\cdot\varphi)(s)ds = \int_{\mathbb{R}} v(s)\mathcal{F}(\varphi)(s)ds. \tag{3.7}$$

By Lemma 3.2 we have $\overline{u} \in C^{\alpha,\beta}(\mathbb{R};X)$ and $D^\beta\overline{u}(t) = v(t) + y_0$, where $y_0 \in X$.

Observe that $\overline{u}(t) \in D(A)$ and $\mathcal{F}(\phi\cdot N)(s)x \in D(A)$ for all $x \in X$, $\phi \in C_c^\infty(\mathbb{R}\setminus\{0\})$. Now, choosing $\phi = a_\cdot\cdot\varphi$ in (3.4) we have from (3.6)

$$\int_{\mathbb{R}} A\overline{u}(s)\mathcal{F}(a_s\cdot\varphi)(s)ds = \int_{\mathbb{R}} w(s)\mathcal{F}(\varphi)(s)ds. \tag{3.8}$$

From Lemma 3.3 we obtain $w(t) = (a * A\overline{u})(t) + y_1$, where $y_1 \in X$. Observe that (3.4), (3.5),(3.6),(3.7) and the identity, $(it)^\beta N(t) = (1 + a_t)AN(t) + I$ imply

$$\begin{aligned}
\int_{\mathbb{R}} v(s)\mathcal{F}(\varphi)(s)ds &= \int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(\mathrm{id}^\beta \cdot \varphi)(s)ds \\
&= \int_{\mathbb{R}} \mathcal{F}(\mathrm{id}^\beta \cdot \varphi \cdot N)(s)f(s)ds \\
&= \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot [I + (1 + a_s)AN])(s)f(s)ds \\
&= \int_{\mathbb{R}} \mathcal{F}(\varphi)(s)f(s)ds + A\int_{\mathbb{R}} \mathcal{F}(\varphi \cdot N)(s)f(s)ds + \\
&\quad \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot T)(s)f(s)ds \\
&= \int_{\mathbb{R}} \mathcal{F}(\varphi)(s)f(s)ds + A\int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(\varphi)(s)ds + \\
&\quad \int_{\mathbb{R}} w(s)\mathcal{F}(\varphi)(s)ds.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} v(s)\mathcal{F}(\varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\varphi)(s)f(s)ds + A\int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(\varphi)(s)ds + \int_{\mathbb{R}} w(s)\mathcal{F}(\varphi)(s)ds. \tag{3.9}$$

It follows from (3.9) and Lemma 2.6 that $v(t) = f(t) + A\overline{u}(t) + w(t) + y_2$ where $y_2 \in X$, and therefore $D^\beta\overline{u}(t) = A\overline{u}(t) + (a * A\overline{u})(t) + f(t) + y$, where $y = y_0 + y_1 + y_2$. Let $u(t) = \overline{u}(t) + x$ where $x = [(a_0 + 1)A]^{-1}y$. Note that x is well defined since $\frac{(i\eta)^\beta}{1+a_\eta} \in \rho(A)$ for all $\eta \in \mathbb{R}$. We observe that u is a solution of (3.1). In fact, since the fractional derivative (in the sense of Caputo) of a constant is zero, we have

$$\begin{aligned}
D^\beta u(t) &= D^\beta \overline{u}(t) \\
&= Au(t) + (a * Au)(t) + f(t) - (a_0 + 1)Ax + y \\
&= Au(t) + (a * Au)(t) + f(t).
\end{aligned}$$

On the other hand, since $\overline{u} \in C^\alpha(\mathbb{R}; [D(A)]) \cap C^{\alpha,\beta}(\mathbb{R}; X)$ we conclude that $u \in C^\alpha(\mathbb{R}; [D(A)]) \cap C^{\alpha,\beta}(\mathbb{R}; X)$ and therefore, u is a solution of equation (3.1).

To see the uniqueness, suppose that

$$D^\beta u(t) = Au(t) + (a * Au)(t), \quad t \in \mathbb{R}. \tag{3.10}$$

As in [30, Appendix A], for $\sigma > 0$, we denote $L_\sigma(u)(\rho)$ by $L_\sigma(u)(\rho) := \hat{u}(\sigma + i\rho) - \hat{u}(-\sigma + i\rho)$, where $\rho \in \mathbb{R}$. Take L_σ in (3.10). From [30, Proposition A.2.(iv)], we have

$$L_\sigma(D^\beta u)(\rho) = A\hat{a}(\sigma + i\rho)L_\sigma(u)(\rho) + G_a^{Au}(\sigma, \rho) + AL_\sigma(u)(\rho), \tag{3.11}$$

with

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}} G_a^{Au}(\sigma,\rho)\phi(\rho)d\rho = 0,$$

for all $\phi\in\mathcal{S}(\mathbb{R})$, where

$$\begin{aligned} G_a^{Au}(\sigma,\rho) &= \int_{-\infty}^{0}\left(\int_{-s}^{\infty} a(\tau)e^{-(\sigma+i\rho)\tau}d\tau\right)e^{-(\sigma+i\rho)s}Au(s)ds \\ &+ \int_{-\infty}^{0}\left(\int_{0}^{-s} a(\tau)e^{(\sigma-i\rho)(s+\tau)}d\tau\right)Au(s)ds \\ &- \int_{-\infty}^{0}\left(\int_{0}^{\infty} a(\tau)e^{-(\sigma+i\rho)\tau}d\tau\right)e^{(\sigma-i\rho)s}Au(s)ds. \end{aligned}$$

A simply computation, shows that the Carleman transform of fractional derivative of u satisfies

$$\widehat{D^\beta u}(\lambda) = \lambda^\beta\hat{u}(\lambda) - \sum_{k=0}^{n-1} u^{(k)}(0)\lambda^{\beta-1-k},\ \text{for } \mathrm{Re}\lambda\neq 0,\ n=\lceil\beta\rceil.$$

Hence,

$$L_\sigma(D^\beta u)(\rho) = (\sigma+i\rho)^\beta L_\sigma(u)(\rho) + c_\beta(\sigma,\rho)\hat{u}(-\sigma+i\rho) - \sum_{k=0}^{n-1} c_{\beta-1-k}(\sigma,\rho)u^{(k)}(0),$$

where $c_\beta(\sigma,\rho) = (\sigma+i\rho)^\beta - (-\sigma+i\rho)^\beta$. Denote $H_\beta(\sigma,\rho)$ by

$$H_\beta(\sigma,\rho) := c_\beta(\sigma,\rho)\hat{u}(-\sigma+i\rho) - \sum_{k=0}^{n-1} c_{\beta-1-k}(\sigma,\rho)u^{(k)}(0).$$

From (3.11) we have

$$\left((\sigma+i\rho)^\beta - (1+\hat{a}(\sigma+i\rho))A\right)L_\sigma(u)(\rho) = G_a^{Au}(\sigma,\rho) - H_\beta(\sigma,\rho).$$

Since $\frac{(i\eta)^\beta}{1+a_\eta}\in\rho(A)$ for all $\eta\in\mathbb{R}$, we obtain,

$$\begin{aligned} L_\sigma(u)(\rho) &= G_a^{Au}(\sigma,\rho)R_\rho - H_\beta(\sigma,\rho)R_\rho - \\ &\quad (\hat{a}(i\rho) - \hat{a}(\sigma+i\rho))AR_\rho L_\sigma(u)(\rho) - \\ &\quad \left((\sigma+i\rho)^\beta - (i\rho)^\beta\right)R_\rho L_\sigma(u)(\rho), \end{aligned}$$

where R_ρ denotes $R_\rho = ((i\rho)^\beta - (1+\hat{a}(i\rho))A)^{-1}$.

A similar argument to used in [30, Lemma A.4] shows that

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}} (\hat{a}(i\rho) - \hat{a}(\sigma+i\rho))AR_\rho L_\sigma(u)(\rho)\phi(\rho)d\rho = 0,$$

for all $\phi\in\mathcal{S}(\mathbb{R})$. Applying the dominated convergence theorem, we have

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}} G_a^{Au}(\sigma,\rho)R_\rho\phi(\rho)d\rho = 0,$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. As in [30, Lemma A.4], we can show that

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}} H_\beta(\sigma,\rho)R_\rho\phi(\rho)d\rho = 0,$$

and

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}} \left((\sigma+i\rho)^\beta-(i\rho)^\beta\right)R_\rho\phi(\rho)d\rho = 0,$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. Therefore, by [30, Proposition A.2.(i)] we obtain

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}} L_\sigma(u)(\rho)\phi(\rho)d\rho = \int_{\mathbb{R}} u(\rho)\mathcal{F}(\phi)(\rho)d\rho = 0,$$

for all $\phi \in \mathcal{S}(\mathbb{R})$. We conclude from Lemma 2.6 that u is constant, that is, $u(t) = x$ for all $t \in \mathbb{R}$ and some $x \in X$. We claim that $x = 0$. In fact, since that u is solution of (3.10) we have

$$0 = D^\beta u(t) = Ax + (a \ast Ax)(t) = (1+a_0)Ax.$$

Since $\frac{(i\eta)^\beta}{1+a_\eta} \in \rho(A)$ for all $\eta \in \mathbb{R}$, we conclude that $x = 0$ and therefore $u \equiv 0$. □

Remark 3.8. *When the underlying Banach space X is B-convex, we may replace the assumption that the kernel a is* 2*-regular in Theorem 3.7, by the weaker condition that a is an* 1*-regular kernel. This follows from Remark 2.3 and the proof of Theorem 3.7.*

We notice that, on periodic vector-valued Lebesgue spaces, $L^p_{2\pi}(\mathbb{R};X)$, $1 < p < \infty$, (where X is a UMD space) and in the scale of periodic Besov spaces $B^s_{p,q}([0,2\pi];X)$, analogous results to Theorem 3.7 have been obtained [11, 28].

Corollary 3.9. *In the context of Theorem 3.7, if condition (ii) is fulfilled, we have that $D^\beta u$, $a \ast Au, Au \in C^\alpha(\mathbb{R};X)$. Moreover, there exists a constant $C > 0$ independent of $f \in C^\alpha(\mathbb{R};X)$ such that*

$$\|D^\beta u\|_{C^\alpha} + \|a \ast Au\|_{C^\alpha} + \|Au\|_{C^\alpha} \le C\|f\|_{C^\alpha}. \tag{3.12}$$

Remark 3.10.

The inequality (3.12) is a consequence of the closed graph theorem and known as the *maximal regularity property* for equation (3.1).

We deduce that the operator S defined by:

$$(Su)(t) := D^\beta u(t) - Au(t) - \int_{-\infty}^t a(t-s)Au(s)ds, \quad t \in \mathbb{R},$$

with domain

$$D(S) = C^{\alpha,\beta}(\mathbb{R};X) \cap C^\alpha(\mathbb{R};[D(A)]),$$

is an isomorphism onto. In fact, by Remark 3.5 we have that the space $H := C^\alpha(\mathbb{R};[D(A)]) \cap C^{\alpha,\beta}(\mathbb{R};X)$ becomes a Banach space under the norm

$$\|u\|_H := \|D^\beta u\|_{C^\alpha} + \|Au\|_{C^\alpha} + \|a \ast Au\|_{C^\alpha} + \|u\|_{C^\alpha}.$$

We remark that such isomorphisms are crucial for the handling of nonlinear evolution equations (see [2]). Indeed, assume that X is a Banach space and A satisfy the condition (ii) in Theorem 3.7. Consider the semilinear problem

$$D^\beta u(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)ds + f(t,u(t)), \quad t \in \mathbb{R}. \tag{3.13}$$

Define the Nemytskii's operator $N : H \to C^\alpha(\mathbb{R};X)$ given by $N(v)(t) = f(t,v(t))$ and the bounded linear operator

$$T := S^{-1} : C^\alpha(\mathbb{R};X) \to H$$

by $T(g) = u$ where u is the unique solution to linear problem

$$D^\beta u(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)ds + g(t).$$

Then, to solve (3.13) we need to show that the operator $R : H \to H$ defined by $R = TN$ has a fixed point. For more details, we refer to Amann [2, 3].

4. Well-posedness of a particular abstract equation

In this section, we consider the following equation

$$D^\beta u(t) + A^\gamma u(t) = f(t), \quad t \in \mathbb{R}, \tag{4.14}$$

where A is a sectorial operator, $0 < \beta < 2$, and $\gamma > 0$. The well-posedness to this class of equations have been studied in [15] and [29].

We begin with some preliminaries on sectorial operators. We recall that a closed, densely defined operator A is *sectorial of angle* $\delta \in (0,\pi)$ if $\sigma(A) \subset \overline{\Sigma}_\delta$, and for every $\delta' \in (\delta,\pi)$

$$\sup_{z \in \mathbb{C}\setminus\Sigma_{\delta'}} \|z(z-A)^{-1}\| < \infty,$$

where $\Sigma_\delta := \{z \in \mathbb{C} : |\arg z| < \delta\}$. For a sectorial operator, define the *sectorial angle* $\omega(A)$ by

$$\omega(A) := \inf\{\delta \in (0,\pi) : A \text{ is sectorial of angle } \delta\}.$$

For every $\delta \in (0,\pi)$ we put

$$H^\infty(\Sigma_\delta) := \{f : \Sigma_\delta \to \mathbb{C} : \|f\|_\infty < \infty\},$$

$$H_0^\infty(\Sigma_\delta) := \left\{ f \in H^\infty(\Sigma_\delta) : \exists \varepsilon > 0 \text{ such that } \sup_{z\in\Sigma_\delta} |f(z)| \left|\frac{1+z^2}{z}\right|^\varepsilon < \infty \right\}.$$

If A is a sectorial operator of angle $\delta \in (0,\pi)$, then

$$\Phi_A(f) := f(A) := \frac{1}{2\pi i}\int_{\partial\Sigma_{\delta'}} f(z)(z-A)^{-1}dz$$

defines a functional calculus from $H_0^\infty(\Sigma_{\delta'})$ into $\mathcal{B}(X)$ for every $\delta' > \delta$. This functional calculus may be extended in a natural way in order to define the fractional powers A^ε for every $\varepsilon > 0$, see [25, 35].

A sectorial operator A admits a *bounded H^∞ functional calculus of angle* $\delta \in [\omega(A), \pi)$ if the functional calculus on $H_0^\infty(\Sigma_{\delta'})$ extends to a bounded linear operator on $H^\infty(\Sigma_{\delta'})$ for every $\delta' \in (\delta, \pi)$.

The well-known examples for general classes of closed linear operator with a bounded H^∞ calculus are

1. normal sectorial operators in a Hilbert space;
2. m-accreative operators in a Hilbert space;
3. generators of bounded C_0-groups on L_p-spaces;
4. negative generators of positive contraction semigroups on L_p-spaces.

The main result of this section is the following.

Theorem 4.11. *Let A be a sectorial operator which admits a bounded H^∞ functional calculus of angle $\omega \in \left(0, \frac{\pi}{\gamma}(1-\frac{\beta}{2})\right)$ on a Banach space X, where $0 < \beta < 2$. If $0 \in \rho(A)$, then (4.14) C^α-well posed.*

Proof. Follow the same lines of [29, Theorem 4.6]. Since $\omega \in \left(0, \frac{\pi}{\gamma}(1-\frac{\beta}{2})\right)$, there exists $\delta > \omega$ such that $\delta < \frac{\pi}{\gamma}(1-\frac{\beta}{2})$. For each $z \in \Sigma_\delta$ and $t \in \mathbb{R}$, define $F(it, z) := (it)^\beta((it)^\beta + z^\gamma)^{-1}$. Note that $\frac{z^\gamma}{(it)^\beta}$ belongs to the sector $\Sigma_{\frac{\pi\beta}{2}+\delta\gamma}$, where $\frac{\pi\beta}{2} + \delta\gamma < \pi$. Hence the distance from the sector $\Sigma_{\frac{\pi\beta}{2}+\delta\gamma}$, to -1 is always positive. Therefore, there exists a constant $M > 0$ independent of $t \in \mathbb{R}$ and $z \in \Sigma_\delta$, such that

$$|F(it, z)| = \left| \frac{1}{1 + \frac{z^\gamma}{(it)^\beta}} \right| \leq M.$$

Since A is invertible and admits a H^∞ functional calculus, the operators $((it)^\beta + A^\gamma)^{-1}$ exist for all $t \in \mathbb{R}$. Thus, by Theorem 3.7 with $a \equiv 0$, we have that the equation (4.14) is C^α-well posed. $\square$

We recall that a linear operator A defined on X is called *non-negative* if $(-\infty, 0) \in \rho(A)$ and and there exists $M > 0$ such that

$$\|\lambda(\lambda - A)^{-1}\| \leq M, \quad \text{for all} \quad \lambda < 0,$$

and A is said to be *positive* if it is non-negative and if, in addition, $0 \in \rho(A)$. See [35] for further details.

Since each self-adjoint, positive operator admits a bounded H^∞ calculus of angle 0, we obtain the following Corollary.

Corollary 4.12. *Let A be a selfadjoint, positive operator defined on a Hilbert space H and $0 < \beta < 2$. Then for every $f \in C^\alpha(\mathbb{R}; H)$ there exists a unique $u \in C^\alpha(\mathbb{R}; [D(A)]) \cap C^{\alpha,\beta}(\mathbb{R}; H)$ such that (4.14) holds for all $t \in \mathbb{R}$.*

5. Examples

We conclude the paper, with some applications of the previous results.

Example 5.13.

If $A = \rho I$, where $\rho \in \mathbb{R} \setminus \{0\}$, and $a \equiv 0$ in equation (3.1) we obtain

$$D^\beta u(t) = \rho u(t) + f(t), \quad t \in \mathbb{R}. \tag{5.15}$$

Clearly, $(i\eta)^\beta \in \rho(A)$ for all $\eta \in \mathbb{R}$ and

$$\sup_{\eta\in\mathbb{R}} \|((i\eta)^\beta - \rho I)^{-1}\| < \infty.$$

Hence, by Theorem 3.7 the equation (5.15) is C^α-well posed. Observe that the solution to (5.15) is given by

$$u(t) = \int_{-\infty}^{t} (t-s)^{\beta-1} E_{\beta,\beta}(\rho(t-s)^\beta) f(s) ds, \quad t \in \mathbb{R}, \tag{5.16}$$

where, for $a, b > 0$, $E_{a,b}$ denotes the Mittag-Leffler function

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b + ak)}, \quad z \in \mathbb{C}.$$

In fact, we notice that the Fourier transform of (5.16) is given by the product of the Laplace transform of $t^{\beta-1}E_{\beta,\beta}(\rho t^\beta)$ (evaluated in the imaginary axis) and the Fourier transform of f. Then, an easy computation, shows that it coincides with the Fourier transform of the given equation (5.15). Observe that by Corollary 3.9, u and $D^\beta u$ belong to $C^\alpha(\mathbb{R}; X)$.

Example 5.14.

For $0 < \beta < 1$, $\gamma > 0$ and $0 \le \delta \le 1$ consider the problem

$$\begin{cases} D^\beta u(t,x) = \Delta u(t,x) + \delta \displaystyle\int_{-\infty}^{t} e^{-\gamma(t-s)} \Delta u(s,x) ds + f(t,x), \quad t \in \mathbb{R}, \\ \qquad u = 0 \quad \text{in } \mathbb{R} \times \partial\Omega, \end{cases} \tag{5.17}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. This class of fractional diffusion equation has been introduced in physics by Nigmatullin [37] to describe diffusion in special types of porous media. Clearly, the kernel a is 2-regular (see [40, Proposition 3.3]), $a_\eta = \tilde{a}(\eta) = \frac{\delta}{i\eta+\gamma}$ and $\frac{(i\eta)^\beta}{1+a_\eta} \in \rho(A)$ for all $\eta \in \mathbb{R}$. Let $\alpha_\eta = \frac{\gamma\delta}{\gamma^2+\eta^2}$ and $\beta_\eta = \frac{-\eta\delta}{\gamma^2+\eta^2}$ the real and imaginary part of a_η, respectively. We recall that, if we take $X = H^{-1}(\Omega)$, then by [24, p. 74], there exists a constant $c > 0$ such that

$$\|(zI - \Delta)^{-1}\| \le \frac{c}{1 + |z|}$$

whenever $\mathrm{Re}z \geq -c(1+|\mathrm{Im}z|)$. Thus, we have that in $X = H^{-1}(\Omega)$, the inequality

$$\|(i\eta)^\beta((i\eta)^\beta I - (1+a_\eta)\Delta)^{-1}\| = \frac{|i\eta|^\beta}{|1+a_t|}\left\|\left(\frac{(i\eta)^\beta}{1+a_\eta}I - \Delta\right)^{-1}\right\| \leq c \qquad (5.18)$$

holds if

$$\mathrm{Re}\left(\frac{(i\eta)^\beta}{1+a_\eta}\right) \geq -c\left(1+\left|\mathrm{Im}\left(\frac{(i\eta)^\beta}{1+a_\eta}\right)\right|\right), \qquad (5.19)$$

for all $\eta \in \mathbb{R}$ and where $c > 0$ is a suitable constant. Since $(i\eta)^\beta = |\eta|^\beta e^{\frac{\pi\beta i}{2}\mathrm{sgn}(\eta)}$ (where $\mathrm{sgn}(\eta)$ denotes the sign of η) we have that (5.19) is equivalent to

$$\begin{aligned} |\eta|^\beta\left[(1+\alpha_\eta)\cos(d_\eta) + \beta_\eta\sin(d_\eta)\right] \quad &\geq \quad -c\big((1+\alpha_\eta)^2 + \beta_\eta^2 + \\ &\qquad |\eta|^\beta|(1+\alpha_\eta)\sin(d_\eta) - \beta_\eta\cos(d_\eta)|\big), \end{aligned} \qquad (5.20)$$

where $d_\eta = \frac{\pi}{2}\beta\mathrm{sgn}(\eta)$. Since $0 < \beta < 1$ we have that $\cos(d_\eta) \geq 0$. Now, observe that if $\eta \geq 0$, then $\beta_\eta \leq 0$ and $\sin(d_\eta) > 0$, therefore $\beta_\eta \sin(d_\eta) \leq 0$. Similarly, $\beta_\eta \sin(d_\eta) \leq 0$ for $\eta < 0$. Thus, the inequality (5.20) holds, in particular for $c = 1$. Hence, in $X = H^{-1}(\Omega)$, we obtain from (5.18) and Theorem 3.7 that the problem (5.17) is C^α well-posed. Moreover, the solution u of (5.17) satisfies $D^\beta u, \Delta u \in C^\alpha(\mathbb{R}; H^{-1}(\Omega))$. In particular, if $\delta = 0$ we obtain that, the fractional diffusion equation

$$\begin{cases} D^\beta u(t,x) = \Delta u(t,x) + f(t,x), & (t,x) \in \mathbb{R}\times\Omega \\ \qquad u = 0 \quad \text{in } \mathbb{R}\times\partial\Omega, \end{cases}$$

is C^α-well posed.

Example 5.15.

Consider the problem

$$\begin{cases} D^\beta u(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) + f(x,t) \\ u(0,t) = u(\pi,t) = 0, \end{cases} \qquad (5.21)$$

with $x \in [0,\pi], t \in \mathbb{R}$ and $\beta > 0$. Let $X = L^2[0,\pi]$ and define $A := \frac{\partial^2}{\partial x^2}$, with domain $D(A) = \{g \in H^2[0,\pi] : g(0) = g(\pi) = 0\}$. It is well known that A generates an analytic C_0- semigroup $T(t)$ on X and $\sigma(A) = \{-n^2 : n \in \mathbb{N}\}$.

If $\beta = 2$, then $(i\eta)^\beta \notin \rho(A)$ for all $\eta \in \mathbb{R}$ and therefore, by Theorem 3.7 the problem (5.21) is not C^α-well posed.

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