Mild solutions to integro-differential equations in Banach spaces

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Abstract

In this paper we study the existence and uniqueness of mild solutions to integro-differential equations in terms of a resolvent operator on the interval $[0, 2\pi]$ and on the real line. Moreover, we characterize the spectrum of the resolvent family that solves the Volterra equation $u' = Au + (a * Au) + f$ in terms of their mild periodic solutions.

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1. Introduction

In the classical theory of heat conduction in homogeneous and isotropic media, the temperature $u(x, t)$ of a point $x \in \Omega \subset \mathbb{R}^n$ $(n = 1, 2, 3)$ at time $t > 0$ satisfies

$$u_t(x, t) = \lambda \Delta u(x, t), \tag{1.1}$$

where Δ is the Laplacian and $\lambda > 0$ is the thermal diffusion coefficient. The equation (1.1) describes sufficiently well the behavior of the temperature in different types of homogeneous materials. However, in other type of materials, such as in materials with fading memory, this description is not satisfactory because in equation (1.1) is assumed that the thermal disturbance at any point in the media is felt instantly at every other point, which is not true in this kind of materials.

The problem of the heat flux in materials with memory was firstly discussed by Coleman and Gurtin [13], Gurtin and Pipkin [14], and Nunziato [31] among others. After a linearization, the authors in [13] consider that the density $e(x, t)$ of the internal energy and the heat flux $\mathbf{q}$ are related by

$$e(x, t) = \nu u(x, t) + \int_{-\infty}^{t} b(t - s) u(x, s) ds, \quad \text{and}$$

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$$\mathbf{q}(x,t) = -\int_{-\infty}^{t} a(t-s)\nabla u(x,s)ds,$$

where $\nu \neq 0$ is a constant, known as the heat capacity, and the positive functions a, b are the *relaxation functions.* Moreover, the heat flux can be considered as a perturbation of the Fourier law, that is,

$$\mathbf{q}(x,t) = -\gamma\nabla u(x,t) - \int_{-\infty}^{t} a(t-s)\nabla u(x,s)ds, \tag{1.2}$$

where $\gamma > 0$ is the constant of thermal conduction. The heat relaxation function a is assumed to be in $L^1(\mathbb{R}_+)$. A typical choice of a is $a(t) = \alpha\frac{t^{\mu-1}}{\Gamma(\mu)}e^{-\beta t}$, where $\alpha > 0, \beta \geq 0$ and $\mu > 0$, see [37, Chapter I, Section 5] and the references therein. With these equations the heat equation with memory reads

$$\nu\partial_t u(x,t) = \gamma\Delta u(x,t) + \int_{-\infty}^{t} a(t-s)\Delta u(x,s)ds + F(x,t), \quad t \in \mathbb{R},$$

where $F(x,t)$ is an appropriate function. This equation can be written in the abstract form

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad t \in \mathbb{R}, \tag{1.3}$$

where $A : D(A) \subset X \to X$ is a closed linear operator defined in a Banach space X, $a \in L^1(\mathbb{R}_+)$ and f is a suitable function.

On the other hand, the Volterra equation

$$u'(t) = Au(t) + \int_{0}^{t} a(t-s)Au(s)ds + f(t), \quad t \in [0,l] \tag{1.4}$$

where A, a and f are defined as before and $l > 0$, also describes several problems in mathematical physics and biology such as electrodynamics with memory, population models, among others.

Integro-differential equations in the form of (1.3) and (1.4) arise in several applied fields, like viscoelasticity or heat conduction with memory. In such applications the operator A is typically the Laplacian, the elasticity operator or the Stokes operator, among others, see for instance [37].

We observe that if $a \equiv 0$, then the equations (1.3) and (1.4) read

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{1.5}$$

and

$$u'(t) = Au(t) + f(t), \quad t \in [0,l], \tag{1.6}$$

respectively. The problem of the existence and uniqueness of mild solutions to equations (1.5) and (1.6) has been considered by several authors in the last years.

See for instance [2, 5, 22, 36, 39] and the references therein. More specifically, a function u is called a mild solution to (1.5) if there exists $y \in X$ such that

$$u(t) = y + A \int_0^t u(s)ds + \int_0^t f(s)ds, \quad t \in \mathbb{R}.$$

In [39, Theorem 1.1] the author shows that if for any $f \in BUC(\mathbb{R}, X)$ (where $BUC(\mathbb{R}, X)$ denotes the space of all bounded and uniformly continuous functions on $\mathbb{R}$ with values in X) the equation (1.5) has a unique mild solution, then $i\eta \in \rho(A)$ for all $\eta \in \mathbb{R}$ and there exists a constant $C > 0$ such that $\|(i\eta - A)^{-1}\| \leq C$, for all $\eta \in \mathbb{R}$. In the case of the $L^p(\mathbb{R}, X)$ space (for $1 \leq p < \infty$) the author in [5] shows that the equation (1.5) has a unique mild solution if and only if $i\eta \in \rho(A)$ for all $\eta \in \mathbb{R}$ and $\{(i\eta - A)^{-1}\}_{\eta\in\mathbb{R}}$ defines an L^p-multiplier in $L^p(\mathbb{R}, X)$. See also [7] for the second order problem.

On the other hand, if $l = 2\pi$ in equation (1.6), then a function $u \in L^p([0, 2\pi], X)$ (where $L^p([0, 2\pi], X)$ denotes the space of all 2π-periodic and p-integrable functions with values in X) is called a 2π-periodic mild solution to (1.6) if $u(0) = u(2\pi)$ and

$$u(t) = u(0) + A \int_0^t u(s)ds + \int_0^t f(s)ds, \quad t \in [0, 2\pi].$$

In this case, the authors in [2] and [22] show, by using some results on vector-valued Fourier multipliers, that equation (1.6) has a unique mild solution for all $f \in L^p([0, 2\pi], X)$ if and only if $ik \in \rho(A)$ for all $k \in \mathbb{Z}$ and $\{(ik - A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier in $L^p([0, 2\pi], X)$. Similar results hold for the second order problem [22] and for the fractional order problem in [6, 35].

Now, if we assume that A is the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$, then the mild solution to (1.5) and (1.6) are given, respectively by

$$u(t) = \int_{-\infty}^t T(t-s)f(s)ds, \quad t \in \mathbb{R}. \tag{1.7}$$

and

$$u(t) = T(t)u(0) + \int_0^t T(t-s)f(s)ds, \quad t \in [0, l] \tag{1.8}$$

In the space $L^p(\mathbb{R}, X)$ the author in [30] shows that if A generates a C_0-semigroup and (1.5) has a unique mild solution for every $f \in L^p(\mathbb{R}, X)$, where $1 \leq p < \infty$, then $i\eta \in \rho(A)$ for all $\eta \in \mathbb{R}$ and there exists a constant $C > 0$ such that $\|(i\eta - A)^{-1}\| \leq \frac{C}{1+|\eta|}$ for all $\eta \in \mathbb{R}$.

On the other hand, in [36] the author shows that if A generates a C_0-semigroup, then the equation (1.6) has a unique 2π-periodic mild solution for all $f \in C([0, 2\pi], X)$ if and only if $1 \in \rho(T(2\pi))$, where $\rho(T(2\pi))$ denotes the resolvent set of the operator $T(2\pi)$. Similar results hold for the second order problem [38] and for the equation of higher order [1, 25].

We remark that in the above mentioned papers, the authors study the existence of mild solutions to the first, second and fractional order problem on the interval $[0, 2\pi]$ or on the real line. The problem of the well-posedness of abstract integro-differential equations, in the sense of the existence of a unique strong solution, has been studied by several authors in the last years, see for instance [2, 8, 9, 10, 19, 20, 21, 29, 32, 33, 34] and the references therein. However, to the best of our knowledge, the problem of the existence and uniqueness of mild solution to integro-differential equations in the form of (1.3) and (1.4) in terms of some resolvent family (in the sense that the mild solution can be written by using this resolvent family) has not been considered in the existing literature.

In this paper we are able to give several results on the existence of mild solutions to the integro-differential equations (1.3) and (1.4). More specifically we give, among others, the following results

1. If for all $f \in L^p([0, 2\pi], X)$ there exists a unique mild solution to (1.3), then $ik \in \rho_a(A)$ for all $k \in \mathbb{Z}$ and $\{(ik - (1 + a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. Here the equation (1.3) is considered on $[0, 2\pi]$;
2. If for all $f \in \mathrm{BUC}(\mathbb{R}, X)$ there exists a unique mild solution $u \in \mathrm{BUC}(\mathbb{R}, X)$ to equation (1.3), then $i\eta \in \rho_a(A)$ for all $\eta \in \mathbb{R}$, and there exists a constant M such that $\| [i\eta - (1 + a_\eta)A]^{-1} \| \leq M$ for all $\eta \in \mathbb{R}$, and
3. If A is the generator of a resolvent family $\{R(t)\}_{t\geq 0}$, then $1 \in \rho(R(2\pi))$ if and only if for any $f \in C([0, 2\pi], X)$ the equation (1.4) (with $l = 2\pi$) admits precisely one 2π-periodic mild solution.

Here, for a kernel $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, the resolvent set $\rho_a(A)$ is defined by $\rho_a(A) := \{\lambda \in \mathbb{C} : (\lambda - (1 + \tilde{a}(\lambda))A) : D(A) \to X$ is invertible and $(\lambda - (1 + \tilde{a}(\lambda))A)^{-1} \in \mathcal{B}(X)\}$, where $\tilde{a}$ denotes the Laplace transform of a, $a_k = \tilde{a}(ik)$ (for $k \in \mathbb{Z}$) and $a_\eta = \hat{a}(\eta)$ (for $\eta \in \mathbb{R}$), where $\hat{a}$ denotes the Hilbert transform of $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ defined on $\mathbb{R}$ as $a(t) = 0$, for $t < 0$.

The paper is organized as follows. Section 2 is devoted to the Preliminaries. In Section 3, we study the existence and uniqueness of 2π-periodic mild solutions to the integro-differential equations (1.3) and (1.4). In Section 4 we introduce a concept of mild solution to (1.3) and (1.4) on the real line and we give a necessary condition for existence and uniqueness of such solutions. Finally, in Section 5 we assume that the operator A in equations (1.3) and (1.4) generates a resolvent family $\{R(t)\}_{t\geq 0}$ and we study the existence and uniqueness of mild solutions to equations (1.3) and (1.4).

2. Preliminaries

Let X, Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ the space of all bounded and linear operators from X into Y. If $X = Y$, then we write simply $\mathcal{B}(X)$. Given a closed linear operator A defined on X, $D(A)$ and $\rho(A)$ denote, respectively, its domain and its resolvent set. By $[D(A)]$ we denote the domain of A equipped with the graph norm.

2.1. Preliminaries on L^p-periodic spaces

Now, we recall some preliminaries L^p-periodic space. For $1 \leq p < \infty$, $L^p([0, 2\pi], X)$ denotes the space of all 2π-periodic Bochner measurable and p-integrable X-valued functions. For a function $f \in L^1([0, 2\pi], X)$ we denote by $\hat{f}(k)$, the k-th Fourier coefficient of f, that is

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt,$$

for all $k \in \mathbb{Z}$. Observe that the Fourier coefficients of f determine completely the function f, that is, $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ if and only if $f(t) = 0$ a.e. The space $C([0, 2\pi], X)$ denotes the Banach space of all continuous function on $[0, 2\pi]$ with values in X.

Definition 2.1. *[2] For $1 \leq p < \infty$, we say that a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an L^p-multiplier if, for each $f \in L^p([0, 2\pi], X)$, there exists $u \in L^p([0, 2\pi], Y)$ such that*

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

For further details on Fourier multipliers we refer the reader to [17, 18]. For $j \in \mathbb{N}$, r_j denotes the j-th Rademacher function on $[0, 1]$ i.e. $r_j(t) = \operatorname{sgn}(\sin(2^j \pi t))$, where sgn is the sign function. For $x \in X$, $r_j \otimes x$, denotes the vector valued function $t \mapsto r_j(t)x$.

Definition 2.2. *A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called R-bounded, if there is a constant $C_p > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}, T_j \in \mathcal{T}, x_j \in X, j = 1, ..., N$ the inequality*

$$\left\| \sum_{j=1}^{N} r_j \otimes T_j x_j \right\|_{L^p((0,1),Y)} \leq C_p \left\| \sum_{j=1}^{N} r_j \otimes x_j \right\|_{L^p((0,1),X)} \tag{2.1}$$

is valid.

For more details on R-bounded families of operators we refer to the reader to [16, Section 3], [17, Section 5.3] and [18, Chapter 8]. Now, we recall a class of Banach spaces, the so-called UMD spaces, which share similar properties with Hilbert spaces and include also the L^p-spaces for $1 < p < \infty$. A Banach space X is said to be UMD, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$. Here the Hilbert transform $\mathcal{H}$ of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$(\mathcal{H}f)(t) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y-t|>\varepsilon} \frac{f(y)}{t-y} dy.$$

Some examples of UMD-spaces include Hilbert spaces, Sobolev spaces $W_p^s(\Omega)$ $1 < p < \infty$, Lebesgue spaces $L^p(\Omega, \mu)$, $1 < p < \infty$, $L^p(\Omega, \mu; X)$, $1 < p < \infty$, whether X is a UMD-space. Moreover, a UMD-space is reflexive and therefore, $L^1(\Omega,\mu), L^\infty(\Omega,\mu)$ and the Hölder space $C^s([0,2\pi];X)$ can not be UMD spaces. More information on UMD spaces can be found in [4, 11, 12] and [17, Chapter 4].

For a kernel $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and a function $g \in L^p([0,2\pi],X)$ (extended by periodicity to $\mathbb{R}$) we obtain, under appropriate assumptions on $a(t)$, that for

$$(a\dot{*}g)(t) := \int_{-\infty}^{t} a(t-s)g(s)ds, \tag{2.2}$$

and $k \in \mathbb{Z}$, the Fourier coefficients of $(a\dot{*}g)$ verify $\widehat{(a\dot{*}g)}(k) = \tilde{a}(ik)\hat{g}(k)$, where $\tilde{a}(ik)$ is the Laplace transform of a evaluated at ik (see for instance [19]) In what follows, we use the following notation:

$$a_k := \tilde{a}(ik), \quad k \in \mathbb{Z}, \tag{2.3}$$

and we assume that $a_k \neq 1$ for all $k \in \mathbb{Z}$. Finally, from [23] we recall the concept of 1-regular sequences.

Definition 2.3. *A sequence $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}\setminus\{0\}$ is said to be 1-regular, if the sequence $\left\{k\dfrac{(c_{k+1}-c_k)}{c_k}\right\}_{k\in\mathbb{Z}}$ is bounded.*

2.2. Preliminaries on $C^\alpha(\mathbb{R};X)$ spaces.

Let $0<\alpha<1$ be fixed. We denote by $C^\alpha(\mathbb{R};X)$ the space of all X-valued functions f on $\mathbb{R}$, such that

$$\|f\|_\alpha = \sup_{t\neq s}\frac{\|f(t)-f(s)\|}{|t-s|^\alpha} < \infty.$$

If we define $\|f\|_{C^\alpha} := \|f\|_\alpha + \|f(0)\|$, then $(C^\alpha(\mathbb{R};X), \|\cdot\|_{C^\alpha})$ is a Banach space.

Now, the Fourier transform of a function $f \in L^1(\mathbb{R};X)$ is defined by $\hat{f}(\eta) := \int_{\mathbb{R}} e^{-i\eta t}f(t)dt$, for $\eta \in \mathbb{R}$. If $a \in L^1(\mathbb{R}_+)$, we will always identify a with its extension on $\mathbb{R}$ by letting $a(t) = 0$ for $t<0$. In such way, when $a \in L^1(\mathbb{R}_+)$, the Fourier transform $\hat{a}(\eta)$ makes sense for all $\eta \in \mathbb{R}$.

In what follows, we always assume that $\hat{a}(\eta) \neq -1$, for all $\eta \in \mathbb{R}$, and we use the following notation:

$$a_\eta := \hat{a}(\eta), \quad \eta \in \mathbb{R}. \tag{2.4}$$

Now, we recall the notion of regular kernels (see [37, p. 69]).

Definition 2.4. *Let $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. The kernel $a(t)$ is called k-*regular *if there is a constant $c>0$ such that*

$$|\lambda^n[\tilde{a}(\lambda)]^{(n)}| \leq c|\tilde{a}(\lambda)|, \quad \textit{for all} \quad \mathrm{Re}(\lambda) > 0, 0 \leq n \leq k.$$

We denote by $L^1(\mathbb{R}_+, t^\alpha dt)$ the set of all functions $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that $\int_0^\infty |a(t)|t^\alpha dt < \infty$, where $0 < \alpha < 1$. Then, as a consequence, such a is always in $L^1(\mathbb{R}_+)$. Given a bounded function v and $a \in L^1(\mathbb{R}_+)$, we write

$$(a \ast v)(t) := \int_{-\infty}^{t} a(t-s)v(s)ds = \int_0^\infty a(s)v(t-s)ds. \tag{2.5}$$

Remark 2.5. *We use the same notation in* (2.2) *and* (2.5) *to the integral* $\int_{-\infty}^t a(t-s)g(s)ds$*. Moreover, if* $\eta \in \mathbb{R}$*, then the Fourier transform of* $(a \ast g)$ *is given by* $\widehat{(a \ast g)}(\eta) = a_\eta \hat{g}(\eta)$.

3. Periodic mild solutions

In this section, we study the existence of mild solutions to the equations

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad t \in [0, 2\pi] \\ \\ u(0) = u(2\pi), \end{cases} \tag{3.6}$$

and

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_{0}^{t} a(t-s)Au(s)ds + f(t), \quad t \in [0, 2\pi] \\ \\ u(0) = u(2\pi), \end{cases} \tag{3.7}$$

where A is a closed linear operator defined in a Banach space X, $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $f : [0, 2\pi] \to X$ is a given function.

Given a kernel $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, the resolvent set $\rho_a(A)$ is defined by $\rho_a(A) := \{\lambda \in \mathbb{C} : (\lambda - (1 + \tilde{a}(\lambda))A) : D(A) \to X$ is invertible and $(\lambda - (1 + \tilde{a}(\lambda))A)^{-1} \in \mathcal{B}(X)\}$, where $\tilde{a}$ denotes the Laplace transform of a. Moreover, if $k \in \mathbb{Z}$, then we notice that $ik \in \rho_a(A)$ if and only if $\{ik/(1 + a_k)\}_{k \in \mathbb{R}} \in \rho(A)$.

From [19, Theorem 2.12] we recall that a function u is called a strong solution to equation (3.6) if $u(t) \in D(A)$ and (3.6) holds for all $t \in [0, 2\pi]$ and we have the following result.

Theorem 3.6. *Let* X *be a UMD space and let* $A : D(A) \subset X \to X$ *be a closed linear operator. Assume that the sequence* $\{a_k\}_{k \in \mathbb{Z}}$ *is a 1-regular sequence. Then, the following assertions are equivalent for* $1 < p < \infty$.

(*i*) *For every* $f \in L^p([0, 2\pi]; X)$*, there exists a unique strong* L^p*-solution of* (3.6)*;*

(*ii*) $\{ik\}_{k \in \mathbb{Z}} \subset \rho_a(A)$ *and* $\{ik(ik - (1 + a_k)A)^{-1}\}_{k \in \mathbb{Z}}$ *is an* L^p*-multiplier; and*

(*iii*) $\{ik\}_{k \in \mathbb{Z}} \subset \rho_a(A)$ *and* $\{ik(ik - (1 + a_k)A)^{-1}\}_{k \in \mathbb{Z}}$ *is R-bounded.*

In order to study mild solutions to equation (3.6) and (3.7), we introduce the following notation. The function g_1 is defined by $g_1(t) = 1$ for all $t \in [0, 2\pi]$.

The usual convolution on $[0,2\pi]$ between the functions f and g, denoted by $(f*g)(t)$, is defined by

$$(f*g)(t)=\int_0^t f(t-s)g(s)ds,$$

for all $t\in[0,2\pi]$. Observe that

$$(g_1*f)(t)=\int_0^t f(s)ds$$

for all $t\in[0,2\pi]$.

Definition 3.7. *Let $f\in L^1_{\text{loc}}(\mathbb{R},X)$. A function $u\in C([0,2\pi],X)$ is called a* mild solution *to (3.6) if $u(0)=u(2\pi)$, $(g_1*u)(t)+(g_1*(a\dot{*}u))(t)\in D(A)$, for all $t\in[0,2\pi]$ and*

$$u(t)\quad=\quad u(0)+A[(g_1*u)(t)+(g_1*(a\dot{*}u))(t)]+(g_1*f)(t), \tag{3.8}$$

for all $t\in[0,2\pi]$.

Observe that if $a(t)\equiv 0$, then this concept of mild solution is the same as in the case of the first order problem $u'(t)=Au(t)+f(t)$, see [2].

Lemma 3.8. *Let $a\in L^1(\mathbb{R}_+)$ and $f\in L^1([0,2\pi],X)$. Define the function G^a_f by $G^a_f(t):=(g_1*(a\dot{*}f))(t)$, $t\in[0,2\pi]$. Then, the Fourier coefficients of G^a_f are given by*

$$\widehat{G^a_f}(k)=-\frac{1}{ik}a_0\hat{f}(0)+\frac{1}{ik}a_k\hat{f}(k),\quad k\in\mathbb{Z}\setminus\{0\}.$$

Proof. It follows similarly to [20, Lemma 4.2]. □

As in Theorem 3.6, the next result characterizes the existence of mild solutions in terms of the operator $((ik)-(1+a_k)A)$ for all $k\in\mathbb{Z}$.

Theorem 3.9. *Let $f\in L^1([0,2\pi],X)$ and $u\in C([0,2\pi],X)$. Assume that $\overline{D(A)}=X$. Then u is a mild solution to problem* (3.6) *if and only if*

$$\hat{u}(k)\in D(A)\quad and\quad ((ik)-(1+a_k)A)\hat{u}(k)=\hat{f}(k), \tag{3.9}$$

for all $k\in\mathbb{Z}$.

Proof. Assume that u is a mild solution to (3.6). From [2, Lemma 3.1] and hypothesis it follows that $\hat{u}(k)\in D(A)$ for all $k\in\mathbb{Z}$ and by (3.8) we obtain (with $t=2\pi$) that

$$0=A[(g_1*u)(2\pi)+G^a_u(2\pi)]+(g_1*f)(2\pi)$$

that is

$$0=A[\hat{u}(0)+a_0\hat{u}(0)]+\hat{f}(0). \tag{3.10}$$

Let $w(t) = u(t) - u(0) - (g_1 * f)(t)$. Since $\widehat{(g_1 * f)}(k) = -\frac{1}{ik}\hat{f}(0) + \frac{1}{ik}\hat{f}(k)$, we have

$$\hat{w}(k) = \hat{u}(k) + \frac{1}{ik}\hat{f}(0) - \frac{1}{ik}\hat{f}(k). \tag{3.11}$$

On the other hand,

$$\begin{aligned} \hat{w}(k) &= A[\widehat{(g_1 * u)}(k) + (g_1 \widehat{* (a \dot{*} u)})(k)] \\ &= A\left[-\frac{1}{ik}(\hat{u}(0) + a_0\hat{u}(0)) + \frac{1}{ik}(\hat{u}(k) + a_k\hat{u}(k))\right]. \end{aligned} \tag{3.12}$$

Therefore, (3.10)-(3.12) imply

$$((ik) - (1 + a_k)A)\hat{u}(k) = \hat{f}(k),$$

for all $k \in \mathbb{Z}$.

Conversely, suppose that (3.9) holds for all $k \in \mathbb{Z}$. We shall prove that for all $x^* \in D(A^*)$, where A^* denotes the adjoint operator of A, we have

$$\langle (g_1 * u)(t) + (g_1 * (a \dot{*} u))(t), A^* x^* \rangle = \langle u(t), x^* \rangle - \langle u(0), x^* \rangle - \langle (g_1 * f)(t), x^* \rangle. \tag{3.13}$$

In fact, if $w(t) := \langle u(t) + (a \dot{*} u)(t), A^* x^* \rangle + \langle f(t), x^* \rangle$, then by (3.9) we have

$$\begin{aligned} \hat{w}(k) &= \langle (1 + a_k)\hat{u}(k), A^* x^* \rangle + \langle \hat{f}(k), x^* \rangle \\ &= ik\langle \hat{u}(k), x^* \rangle. \end{aligned}$$

In particular $\hat{w}(0) = 0$. Define the function $v(t) := (g_1 * w)(t) - \langle u(t), x^* \rangle$. Then

$$\hat{v}(k) = \frac{1}{ik}\hat{w}(k) - \langle \hat{u}(k), x^* \rangle = 0,$$

for all $k \in \mathbb{Z} \setminus \{0\}$. Then, the function $v(t)$ is constant, that is $v(t) = v(0) = -\langle u(0), x^* \rangle$, for all $t \in [0, 2\pi]$, which implies (3.13). On the other hand, if $t = 2\pi$ in (3.13) we obtain

$$0 = 2\pi\langle (1 + a_0)A\hat{u}(0) + \hat{f}(0), x^* \rangle = \langle u(2\pi), x^* \rangle - \langle u(0), x^* \rangle$$

for all $x^* \in X^*$ and therefore $u(0) = u(2\pi)$. This finishes the proof of the Theorem. $\square$

We denote by $\sigma_p^a(A) := \left\{\lambda \in \mathbb{C} : \frac{\lambda}{1+\hat{a}(\lambda)} \text{ is an eigenvalue of } A\right\}$. As a consequence of Theorem 3.9, we have the following result.

Corollary 3.10. *The following assertions are equivalent:*

(i) *For all $f \in L^1([0, 2\pi], X)$ there exists at most one mild solution of* (3.6)*; and*

(ii) $i\mathbb{Z} \cap \sigma_p^a(A) = \emptyset$.

Proof. Assume that for all $f \in L^1([0,2\pi],X)$ there exists at most one mild solution u of (3.6). Since $(1+a_k) \neq 0$, the Theorem 3.9 implies (with $f \equiv 0$) that $\hat{u}(k) \in D(A)$ and $(ik/(1+a_k) - A)\,\hat{u}(k) = 0$, for all $k \in \mathbb{Z}$, but the hypothesis implies that $u \equiv 0$, and therefore $i\mathbb{Z} \cap \sigma_p^a(A) = \emptyset$.

Conversely, suppose that $i\mathbb{Z} \cap \sigma_p^a(A) = \emptyset$. Let f be a function in $L^1([0,2\pi],X)$ and assume that u and v are mild solutions to (3.6). By Theorem 3.9 we have $\hat{u}(k), \hat{v}(k) \in D(A)$ and $((ik) - (1+a_k)A)\hat{u}(k) = \hat{f}(k) = ((ik) - (1+a_k)A)\hat{v}(k)$, for all $k \in \mathbb{Z}$. Since $(1+a_k) \neq 0$, we obtain $(ik/(1+a_k) - A)\,\widehat{(u-v)}(k) = 0$, for all $k \in \mathbb{Z}$. The hypothesis implies that $\widehat{(u-v)}(k) = 0$ for all $k \in \mathbb{Z}$, and thus $u(t) = v(t)$ a.e., by the uniqueness of the Fourier coefficients. $\square$

The next Proposition gives a generalization of [2, Proposition 3.4].

Proposition 3.11. *Let X be a Banach space and $1 \leq p < \infty$. Assume that $\overline{D(A)} = X$. If for all $f \in L^p([0,2\pi],X)$ there exists a unique mild solution to (3.6), then $\{ik\}_{k\in\mathbb{Z}} \subset \rho_a(A)$ and $\{(ik - (1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Proof. We first notice that in the proof of $(i) \Rightarrow (ii)$ in Theorem 3.6 the UMD condition on the Banach space X is not necessary. Therefore, we can prove similarly to Theorem 3.6 that $\frac{ik}{1+a_k} \in \rho(A)$ for all $k \in \mathbb{Z}$. Take $f \in L^p([0,2\pi],X)$. Let u be the unique mild solution to (3.6). From (3.9) it follows that $\hat{u}(k) = (ik-(1+a_k)A)^{-1}\hat{f}(k)$ for all $k \in \mathbb{Z}$, which implies that $\{(ik-(1+a_k)A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier by [2, Proposition 1.1]. $\square$

It is not known whether the converse of Proposition 3.11 is true in general: If the converse holds, then for $f, u \in L^p([0,2\pi],X)$ we have $\hat{u}(k) = (ik - (1+a_k)A)^{-1}\hat{f}(k)$ for all $k \in \mathbb{Z}$. However, it is not clear in general whether the function u is continuous.

Now, we study mild solutions to equation (3.7). We first introduce its definition.

Definition 3.12. *Let $f \in L^1_{\text{loc}}(\mathbb{R},X)$. A function $u \in C([0,2\pi],X)$ is called a* mild solution *to (3.7) if $u(0) = u(2\pi)$, $(g_1 * u)(t) + (g_1 * a * u)(t) \in D(A)$, for all $t \in [0,2\pi]$ and*

$$u(t) \quad = \quad u(0) + A[(g_1 * u)(t) + (g_1 * a * u)(t)] + (g_1 * f)(t), \qquad (3.14)$$

for all $t \in [0,2\pi]$.

Theorem 3.13. *Let $f \in L^1([0,2\pi],X)$ and $u \in C([0,2\pi],X)$. Assume that $\overline{D(A)} = X$. Then u is a mild solution to problem (3.7) if and only if*

$$\hat{u}(k) \in D(A) \quad \text{and} \quad ((ik) - \widehat{(a*u)}(k)A) = \hat{f}(k),$$

for all $k \in \mathbb{Z}$.

Proof. If u is a mild solution to (3.7), then taking $t = 2\pi$ in (3.14) we obtain

$$0 = A\left[\int_0^{2\pi} u(s)ds + \int_0^{2\pi} (a * u)(s)ds\right] + \int_0^{2\pi} f(s)ds,$$

which is equivalent to $0 = A[\hat{u}(0) + \widehat{(a * u)}(0)] + \hat{f}(0)$. And then, the proof follows similarly to the proof of Theorem 3.9. We omit the details. $\square$

Remark 3.14. *Since $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ is not a 2π-periodic function, then $\widehat{(a * u)}(k) \neq \hat{a}(k)\hat{u}(k)$. We notice that if $a \in L^1([0, 2\pi], X)$ and we consider the integro-differential equation*

$$u'(t) = Au(t) + \frac{1}{2\pi}\int_0^{2\pi} a(t-s)Au(s)ds + f(t), \quad t \in [0, 2\pi] \tag{3.15}$$

with the periodic conditions $u(0) = u(2\pi)$, then the same method of proof of Theorem 3.13 allows to prove that if $\overline{D(A)} = X$, then u is a mild solution to problem (3.15) *if and only if*

$$\hat{u}(k) \in D(A) \quad \text{and} \quad ((ik) - \hat{a}(k)\hat{u}(k)A) = \hat{f}(k),$$

for all $k \in \mathbb{Z}$.

4. Mild solutions on the real line

In this section we study mild solution to the following equation on the real line

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad t \in \mathbb{R} \tag{4.16}$$

where A is a closed linear operator defined on a Banach space X and $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$. We notice that for $f \in C^\alpha(\mathbb{R}, X)$ we have the following result [20, 34].

Theorem 4.15. *Let $0 < \alpha < 1$. Let X be a Banach space and let $A : D(A) \subset X \to X$ be a closed linear operator. Assume that $a \in L^1(\mathbb{R}_+, t^\alpha dt)$ and is a 2-regular kernel. Then the following assertions are equivalent*

(i) *For every $f \in C^\alpha(\mathbb{R}; X)$, there exists a unique strong C^α-solution of* (4.16)*;*
(ii) *$\{i\eta\}_{\eta \in \mathbb{R}} \subset \rho_a(A)$ and $\sup_{\eta \in \mathbb{R}} \|i\eta(i\eta - (1 + a_\eta)A)^{-1}\| < \infty$.*

In this section, we introduce a concept of *mild solution* to equation (4.16) and we give necessary conditions for its existence and uniqueness.

Now we use the same notation to define the function g_1 on the real line. Thus, $g_1(t) = 1$ for all $t \in \mathbb{R}$. The convolution between the functions f and g, denoted by $(f * g)(t)$, is defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds,$$

for all $t \in \mathbb{R}$. Observe that

$$(g_1 * f)(t) = \int_0^t f(s)ds,$$

for all $t \in \mathbb{R}$. By $\mathrm{BUC}(\mathbb{R}, X)$ we denote the space of all bounded and uniformly continuous functions on $\mathbb{R}$ with values in X equipped with the norm $\| \cdot \|_\infty$.

Definition 4.16. *Let $f \in \mathrm{BUC}(\mathbb{R}, X)$. A function $u \in \mathrm{BUC}(\mathbb{R}, X)$ is called a* mild solution *to (4.16) if $(g_1 * u)(t) + (g_1 * (a \dot{*} u))(t) \in D(A)$, for all $t \in \mathbb{R}$ and there exists $y \in X$ such that*

$$u(t) \quad = \quad y + A[(g_1 * u)(t) + (g_1 * (a \dot{*} u))(t)] + (g_1 * f)(t), \tag{4.17}$$

for all $t \in \mathbb{R}$.

We observe that the vector y in this definition is unique, and that if $a \equiv 0$, then this concept of mild solution coincides with the notion of mild solution to the first order problem on the real line $u'(t) = Au(t) + f(t)$, $t \in \mathbb{R}$, see [3, Chapter 3] and [39].

Now, we consider the problem of the existence and uniqueness of mild solution to equation (4.16) on the real line. On the space $\mathrm{BUC}(\mathbb{R}, X)$ we define the linear operator $\mathcal{L} : \mathrm{BUC}(\mathbb{R}, X) \to \mathrm{BUC}(\mathbb{R}, X)$ which takes a function $f \in \mathrm{BUC}(\mathbb{R}, X)$ into the solution $u \in \mathrm{BUC}(\mathbb{R}, X)$ of equation (4.16). The operator $\mathcal{L}$ is well-defined by [39, Section 1]. If such solution u is unique for each function f, then by the closed graph theorem $\mathcal{L}$ is a bounded operator. Moreover, we notice that if the mild solution u is once differentiable, that is, $u \in C^1(\mathbb{R}, X)$, then u is a classical solution to (4.16).

The next result gives necessary conditions for the existence and uniqueness of mild solutions to (4.16). Its proof follows similarly to [39, Theorem 2.5].

Theorem 4.17. *Let $a \in L^1(\mathbb{R}_+)$. Let $A : D(A) \subset X \to X$ be a closed linear operator defined in a Banach space X. Assume that for every $f \in \mathrm{BUC}(\mathbb{R}, X)$ there exists a unique mild solution $u \in \mathrm{BUC}(\mathbb{R}, X)$ to equation (4.16). Then $i\eta \in \rho_a(A)$ for all $\eta \in \mathbb{R}$, and there exists a positive constant M such that $\| [i\eta - (1 + a_\eta)A]^{-1} \| \leq M$ for all $\eta \in \mathbb{R}$.*

Proof. We first prove that $[i\eta - (1 + a_\eta)A]$ is surjective. We take arbitrary $\eta \in \mathbb{R}$ and $y \in X$. For $s, t \in \mathbb{R}$, we define the function $f_s(t) := e^{i\eta(t+s)}y = e^{i\eta s}f_0(t) = f_0(t + s)$ where $f_0(t) := e^{i\eta t}y$. Since $f_s \in \mathrm{BUC}(\mathbb{R}, X)$ there exists a unique mild solution $u_s \in \mathrm{BUC}(\mathbb{R}, X)$ to (4.16). We claim that

$$u_s(t) = e^{i\eta s}u_0(t) = u_0(s + t) \tag{4.18}$$

for all $s, t \in \mathbb{R}$. In fact, since u_s is a mild solution to equation (4.16) with f_s, there exists $y_s \in X$ such that

$$u_s(t) \quad = \quad y_s + A[(g_1 * u_s)(t) + (g_1 * (a \dot{*} u_s))(t)] + (g_1 * f_s)(t), \tag{4.19}$$

for all $t \in \mathbb{R}$. Multiplying both sides by $e^{-i\eta s}$ we obtain

$$\begin{aligned} e^{-i\eta s}u_s(t) &= e^{-i\eta s}y_s + A\left[\int_0^t e^{-i\eta s}u_s(r)dr + \int_0^t e^{-i\eta s}(a\dot{*}u_s)(r)dr\right] \\ &\quad + \int_0^t e^{-i\eta s}f_s(r)dr. \end{aligned}$$

Then, $e^{-i\eta s}u_s(t)$ is a mild solution to (4.16) with f_0, since

$$e^{-i\eta s}(a\dot{*}u_s)(r) = \int_{-\infty}^{r} a(r-w)e^{-i\eta s}u_s(w)dw \text{ and } \int_0^t e^{-i\eta s}f_s(r)dr = \int_0^t f_0(r)dr.$$

From the uniqueness, we obtain $e^{-i\eta s}u_s(t) = u_0(t)$ for all $s, t \in \mathbb{R}$ and thus we get the first equality in (4.18).

On the other hand, since u_0 is a mild solution of (4.16) with f_0, there exists $y_0 \in X$ such that

$$u_0(t) = y_0 + A[(g_1 * u_0)(t) + (g_1 * (a\dot{*}u_0))(t)] + (g_1 * f_0)(t),$$

for all $t \in \mathbb{R}$. Then

$$u_0(s+t) = y_0 + A[(g_1 * u_0)(s+t) + (g_1 * (a\dot{*}u_0))(s+t)] + (g_1 * f_0)(s+t),$$

and

$$u_0(s) = y_0 + A[(g_1 * u_0)(s) + (g_1 * (a\dot{*}u_0))(s)] + (g_1 * f_0)(s),$$

which implies

$$\begin{aligned} u_0(s+t) - u_0(s) &= A\big[(g_1 * u_0)(s+t) - (g_1 * u_0)(s) \\ &\quad +(g_1 * (a\dot{*}u_0))(s+t) - (g_1 * (a\dot{*}u_0))(s)\big] \\ &\quad +[(g_1 * f_0)(s+t) - (g_1 * f_0)(s)]. \end{aligned} \tag{4.20}$$

From (4.19) and (4.20) we have

$$\begin{aligned} [u_s(t) - u_0(s+t)] &= [y_s - u_0(s)] + A\big[(g_1 * u_s)(t) - (g_1 * u_0)(s+t) \\ &\quad +(g_1 * u_0)(s) + (g_1 * (a\dot{*}u_s))(t) - (g_1 * (a\dot{*}u_0))(s+t) \\ &\quad +(g_1 * (a\dot{*}u_0))(s)\big] \\ &\quad +[(g_1 * f_s)(t) - (g_1 * f_0)(s+t) + (g_1 * f_0)(s)]. \end{aligned} \tag{4.21}$$

Let $U(t) := u_s(t) - u_0(s+t)$. Easy computations show that

$$[(g_1 * u_s)(t) - (g_1 * u_0)(s+t) + (g_1 * u_0)(s)] = (g_1 * U)(t)$$

$$[(g_1 * (a\dot{*}u_s))(t) - (g_1 * (a\dot{*}u_0))(s+t) + (g_1 * (a\dot{*}u_0))(s)] = (g_1 * (a\dot{*}U))(t)$$

and

$$[(g_1 * f_s)(t) - (g_1 * f_0)(s+t) + (g_1 * f_0)(s)] = 0.$$

From (4.21) we obtain

$$U(t) \quad = \quad [y_s - u_0(s)] + A[(g_1 * U)(t) + (g_1 * (a \dot{*} U))(t)].$$

Therefore, U is a mild solution to the homogeneous equation $u'(t) = Au(t) + (a\dot{*}Au)(t)$. By uniqueness, we conclude that $U(t) = 0$ for all $t \in \mathbb{R}$ and therefore $u_s(t) = u_0(s+t)$. The claim is proved.

Now, we take $x = u_0(0)$. By the claim, we have $u_0(t) = u_0(0+t) = u_0(t+0) = e^{i\eta t}u_0(0) = e^{i\eta t}x$, that is, $u_0(t) = e^{i\eta t}x$. Note that $u_0(\cdot) \in C^1(\mathbb{R}, X)$ and therefore u is a classical solution of (4.16) with $f_0(t)$, that is

$$u_0'(t) = Au_0(t) + (a\dot{*}Au_0)(t) + f_0(t)$$

for all $t \in \mathbb{R}$. In particular, if $t = 0$ then $x \in D(A)$ and we obtain

$$[i\eta - (1 + a_\eta)A]x = f_0(0) = y,$$

which implies that $[i\eta - (1 + a_\eta)A]$ is surjective for all $\eta \in \mathbb{R}$.

In order to prove the injectivity, let $\eta \in \mathbb{R}$ and suppose that for $x \in D(A)$

$$[i\eta - (1 + a_\eta)A]x = 0. \tag{4.22}$$

Let $u(t) = e^{i\eta t}x$. Then, u is a classical solution (and then a mild solution) to (4.16) with $f \equiv 0$, because $(a\dot{*}Au)(t) = e^{i\eta t}a_\eta Ax$. From (4.22) we obtain

$$u'(t) - Au(t) - (a\dot{*}Au)(t) = e^{i\eta t}[i\eta - (1 + a_\eta)A]x = 0. \tag{4.23}$$

and from the uniqueness it follows that $u(t) = 0$ for all $t \in \mathbb{R}$ and thus $x = 0$. Therefore, $[i\eta - (1 + a_\eta)A]$ is injective.

Finally, we take arbitrary $\eta \in \mathbb{R}$ and $y \in X$. Define $x := [i\eta - (1+a_\eta)A]^{-1}y$. Then $u_0(t) = e^{i\eta t}x$ is a classical solution to (4.16) with $f_0(t) = e^{i\eta t}y$, since

$$u'(t) - Au(t) - (a\dot{*}Au)(t) = e^{i\eta t}[i\eta - (1 + a_\eta)A]x = e^{i\eta t}y = f_0(t).$$

On the other hand, observe that $\|[i\eta - (1 + a_\eta)A]^{-1}y\|_X = \|x\|_X = \|u_0\|_\infty$ and $\|y\|_X = \|f_0\|_\infty$. Since the linear operator $\mathcal{L}$ is bounded we obtain

$$\|[i\eta - (1 + a_\eta)A]^{-1}y\|_X = \|x\|_X = \|u_0\|_\infty = \|\mathcal{L}f_0\|_\infty \leq \|\mathcal{L}\|\,\|f_0\|_\infty = \|\mathcal{L}\|\,\|y\|_X.$$

Therefore, there exists a constant $M := \|\mathcal{L}\|$ such that

$$\|\,[i\eta - (1 + a_\eta)A]^{-1}\,\| \leq M,$$

for all $\eta \in \mathbb{R}$. Therefore, we conclude that $i\eta \in \rho_a(A)$ for all $\eta \in \mathbb{R}$. $\square$

5. A resolvent family approach

In this section we study mild solutions to equations (3.6), (3.7) and (4.16) in terms of resolvent families. The concept of *resolvent family* was introduced by Da Prato and Ianelli in [15, Definition 1] as an extension of the notion of C_0-semigroups to study the existence of mild solutions to the following integro-differential equations

$$u'(t) = \int_0^t k(t-s)Au(s)ds, \quad u(0) = u_0, \tag{5.24}$$

where $t \geq 0$, A is a closed linear operator defined in a Banach space X, $u_0 \in X$ and $k \in L^1_{\text{loc}}(\mathbb{R}_+)$. The existence of a resolvent family $\{S(t)\}_{t\geq 0}$ to problem (5.24) ensures a solution to the inhomogeneous problem

$$u'(t) = \int_0^t k(t-s)Au(s)ds + f(t), \quad u(0) = u_0, \tag{5.25}$$

for any continuous function f, since in this case the solution to (5.25) is given in terms of its resolvent family as

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds.$$

We notice that the resolvent family $\{S(t)\}_{t\geq 0}$ verifies

$$S(t)x = x + \int_0^t \kappa(t-s)AS(s)xds,$$

for all $x \in X$, where $\kappa(t) = (g_1 * k)(t)$. The theory of resolvent families had a rapid development. For example, the Volterra equation

$$u(t) = f(t) + \int_0^t k(t-s)Au(s)ds \tag{5.26}$$

is well-posed (in the sense that it has a unique solution) if and only if the equation (5.26) admits a resolvent (now also called in the literature as *resolvent family*) $\{U(t)\}_{t\geq 0}$, see for instance [37, Chapter 1]. In this case, the family $\{U(t)\}_{t\geq 0}$ verifies

$$U(t)x = x + \int_0^t k(t-s)AU(s)xds,$$

for all $x \in X$, and if f is differentiable, then the solution to (5.26) is given by

$$u(t) = U(t)f(0) + \int_0^t U(t-s)f'(s)ds.$$

Some other general concepts such as cosine and sine families [3], integrated semigroups [3], α-times resolvent [26], (a,k)-regularized families [27], convoluted semigroups [24], among others, can be considered as resolvent families, since they have an important role in the representation of the solutions to certain abstract integro/differential equations.

Now, we introduce a definition of resolvent family to the equation (4.16).

Definition 5.18. *Let A be closed linear operator with domain $D(A)$ defined in a Banach space X. We say that A is the generator of a* resolvent family, *if there exists $\omega \geq 0$ and a strongly continuous function $R : [0, \infty) \to \mathcal{B}(X)$ such that $\int_0^t R(s)\, ds$ is exponentially bounded, $\left\{ \frac{\lambda}{1+\hat{a}(\lambda)} : \mathrm{Re}\lambda > \omega \right\} \subset \rho(A)$, and for all $x \in X$,*

$$\frac{1}{1+\hat{a}(\lambda)} \left(\frac{\lambda}{1+\hat{a}(\lambda)} - A \right)^{-1} x = \int_0^\infty e^{-\lambda t} R(t) x dt, \ \ \mathrm{Re}\lambda > \omega.$$

In this case, $\{R(t)\}_{t \geq 0}$ is called the resolvent family generated by A.

Comparing Definition 5.18 with the concept of regularized families introduced in [27] we observe that the resolvent $\{R(t)\}_{t\geq 0}$, is in fact a (b, g_1)-regularized family, where $b(t) := g_1(t) + (g_1 * a)(t)$. Moreover, the function $R(t)$ satisfies the following functional equation (see [28]):

$$R(s)(b * R)(t) - (b * R)(s)R(s) = (b * R)(t) - (b * R)(s),$$

for all $t, s \geq 0$, and, if the operator A, with domain $D(A)$, is the infinitesimal generator of a resolvent family $\{R(t)\}_{t\geq 0}$, then for all $x \in D(A)$ we have

$$Ax = \lim_{t \to 0^+} \frac{R(t)x - x}{(g_1 * b)(t)}.$$

For example, if $a(t) = 0$ for all $t \geq 0$, then $R(t)$ corresponds to a C_0-semigroup.

We have also the following result.

Proposition 5.19. *[27] Let $\{R(t)\}_{t\geq 0}$ be a resolvent family generated by A. Then the following holds:*

(i) $R(t)x \in D(A)$ and $R(t)Ax = AR(t)x$ for all $x \in D(A)$ and $t \geq 0$.

(ii) If $x \in D(A)$ and $t \geq 0$, then

$$R(t)x = x + \int_0^t b(t-s)AR(s)xds \tag{5.27}$$

(iii) If $x \in X$ and $t \geq 0$, then $\int_0^t b(t-s)R(s)xds \in D(A)$, and

$$R(t)x = x + A\int_0^t b(t-s)R(s)xds.$$

In particular, $R(0) = I$.

Now, we introduce a different concept of mild solution to (4.16) in case of when A generates a resolvent family.

Definition 5.20. *Let A be the generator of a resolvent family $\{R(t)\}_{t\geq 0}$. We say that $u \in C(\mathbb{R}, X)$ is a mild solution to* (4.16) *if*

$$u(t) = \int_{-\infty}^{t} R(t-s)f(s)ds, \tag{5.28}$$

for all $t \in \mathbb{R}$.

Observe that if $f(t) \in D(A)$ for all $t \in \mathbb{R}$, then u defined in (5.28) is differentiable and

$$u'(t) = (R' \dot{*} f)(t) + f(t), \tag{5.29}$$

because $R(0) = I$. Now, if we integrate (5.29) on the interval $[0,t]$, then we obtain

$$u(t) - u(0) = (g_1 * (R' \dot{*} f))(t) + (g_1 * f)(t).$$

Since $b(0) = 1$ and $b'(t) = a(t)$, from Proposition 5.19 (*ii*) we have $R'(t)x = (b' * AR)(t)x + b(0)AR(t)x = (a * AR)(t)x + AR(t)x$ for all $x \in D(A)$ and $t \geq 0$. Thus

$$\begin{aligned} u(t) &= u(0) + (g_1 * ((a * AR) \dot{*} f))(t) + (g_1 * (AR \dot{*} f))(t) + (g_1 * f)(t) \\ &= u(0) + A[(g_1 * ((a * R) \dot{*} f))(t) + (g_1 * u)(t)] + (g_1 * f)(t). \end{aligned}$$

On the other hand, an easy computation shows that $((a * R) \dot{*} f)(t) = (a \dot{*} u)(t)$. Therefore,

$$u(t) = u(0) + A[(g_1 * (a \dot{*} u))(t) + (g_1 * u)(t)] + (g_1 * f)(t),$$

for all $t \in \mathbb{R}$. We have proved the following Proposition.

Proposition 5.21. *Let A be the generator of a resolvent family $\{R(t)\}_{t\geq 0}$. If $f(t) \in D(A)$ for all $t \in \mathbb{R}$, then the function u defined by* (5.28) *is a mild solution to equation* (4.16) *according to Definition 4.16 with $y = u(0)$.*

Definition 5.22. *Let A be the generator of a resolvent family $\{R(t)\}_{t\geq 0}$. We say that $u \in C([0, 2\pi], X)$ is a mild solution to equation* (3.7) *if $u(0) = u(2\pi)$ and*

$$u(t) = R(t)u(0) + \int_{0}^{t} R(t-s)f(s)ds, \tag{5.30}$$

for all $t \in \mathbb{R}$.

The next result relates the Definitions 5.22 and 3.12 of mild solution to equation (3.7) in the case that A is the generator of a resolvent family.

Proposition 5.23. *Let A be the generator of a resolvent family $\{R(t)\}_{t\geq 0}$. Then, the function u defined by* (5.30) *is a mild solution to equation* (3.7) *if and only if it is a mild solution according to Definition 3.12.*

Proof. Suppose that u defined by (5.30) is a mild solution to equation (3.7) according to Definition 5.22. By Proposition 5.19 (*iii*) we have $(b*R)(t)x \in D(A)$ for all $t \geq 0, x \in X$, and $R(t)x = x + A(b*R)(t)x$ for all $x \in X$. By [3, Proposition 1.1.7], we get $(b*R*f)(t) \in D(A)$ and

$$(R*f)(t) = (g_1*f)+A(b*R*f)(t) = (g_1*f)+A[(g_1*R*f)(t)+(g_1*a*R*f)(t)]. \tag{5.31}$$

Since u verifies (5.30) we obtain

$$(b*u)(t) = (b*R)(t)u(0) + (b*R*f)(t), \tag{5.32}$$

and therefore $(b*u)(t) \in D(A)$ for all $t \geq 0$. By using (5.31)-(5.32) and Proposition 5.19 (*iii*), we have

$$\begin{aligned} A[(g_1*u)(t)+(g_1*a*u)(t)] &= A(b*u)(t) \\ &= A[(b*R)(t)u(0)+(b*R*f)(t)] \\ &= R(t)u(0)-u(0)+(R*f)(t)-(g_1*f)(t) \\ &= u(t)-u(0)-(g_1*f)(t), \end{aligned}$$

that is, u is a mild solution according to Definition 3.12.

Conversely, if u is a mild solution according to Definition 3.12, then $(g_1*u)(t)+(g_1*a*u)(t) = (b*u)(t) \in D(A)$ and $u(t) = u(0)+A(b*u)(t)+(g_1*f)(t)$. Let v be the function defined by

$$v(t) := (g_1*u)(t)-(g_1*R)(t)u(0)-(g_1*R*f)(t). \tag{5.33}$$

Then $(a*v)(t) = (g_1*a*u)(t)-(g_1*a*R)(t)u(0)-(g_1*a*R*f)(t)$ for all t, which implies that

$$\begin{aligned} v(t)+(a*v)(t) &= [(g_1*u)(t)+(g_1*a*u)(t)] \\ &\quad -[(g_1*R)(t)+(g_1*a*R)(t)]u(0) \\ &\quad -[(g_1*R*f)(t)+(g_1*a*R*f)(t)] \\ &= (b*u)(t)-(b*R)(t)u(0)-(b*R*f)(t). \end{aligned}$$

By Proposition 5.19, $(b*R)(t)u(0), (b*R*f)(t) \in D(A)$ and by hypothesis $(b*u)(t) \in D(A)$ and thus $v(t)+(a*v)(t) \in D(A)$ for all $t \geq 0$.

Since $v(t)+(a*v)(t) \in D(A)$ and $b(t) = g_1(t)+(g_1*a)(t)$ we obtain by (5.31) that

$$\begin{aligned} v'(t) &= u(t)-R(t)u(0)-(R*f)(t) \\ &= u(0)+A(b*u)(t)+(g_1*f)(t)-R(t)u(0)-(R*f)(t) \\ &= -A(b*R)(t)u(0)+A(b*u)(t)-A(b*R*f)(t) \\ &= A\big[-(g_1*R)(t)u(0)-(g_1*a*R)(t)u(0)+(g_1*u)(t)+(g_1*a*u)(t) \\ &\quad -(g_1*R*f)(t)-(g_1*a*R*f)(t)\big] \\ &= A[v(t)+(a*v)(t)]. \end{aligned}$$

On the other hand, if $S(t)y := (b * R)(t)y$ for $y \in X$, then $S(t)y \in D(A)$ by Proposition 5.19 and $AS(t)y = R(t)y - y$ for all $y \in X$. Let $w(s) := S(t-s)v(s)$ for $0 \leq s \leq t$. Then $S'(t)y = (b' * R)(t)y + b(0)R(t)y = (a * R)(t)y + R(t)y$ and

$$\begin{aligned} w'(s) &= -S'(t-s)v(s) + S(t-s)v'(s) \\ &= -R(t-s)v(s) - (a * R)(t-s)v(s) + AS(t-s)[v(s) + (a * v)(s)] \\ &= -(a * R)(t-s)v(s) - v(s) + R(t-s)(a * v)(s) - (a * v)(s). \end{aligned}$$

Moreover, $w(t) = S(0)v(t) = 0$ and $w(0) = S(t-0)v(0) = 0$. Thus, $w(t) = w(0) = 0$. Therefore,

$$\begin{aligned} 0 = w(t) &= \int_0^t w'(s)ds \\ &= -((a * R) * v)(t) - (g_1 * v)(t) + (R * (a * v))(t) - (g_1 * (a * v))(t) \\ &= -(g_1 * v)(t) - (g_1 * a * v)(t) \end{aligned}$$

for all $t \geq 0$, which implies that $0 = v(t) + (a * v)(t)$. Since $v'(t) = A[v(t) + (a * v)(t)]$ we conclude that v is constant. But $v(0) = 0$ and thus $v(t) = 0$ for all $t \geq 0$. Differentiating function v in (5.33) we obtain $u(t) = R(t)u(0) + (R * f)(t)$ for all $t \geq 0$. We conclude that u is a mild solution according to Definition 5.22. $\square$

The next result extends [36, Theorem 1] for C_0-semigroups.

Theorem 5.24. *Let X be a Banach space and $\{R(t)\}_{t \geq 0}$ the resolvent family in X generated by A. Then $1 \in \rho(R(2\pi))$ if and only if for any $f \in C([0, 2\pi], X)$ the equation (3.7) admits precisely one 2π-periodic mild solution.*

Proof. We first assume that $1 \in \rho(R(2\pi))$. Let $f \in C([0, 2\pi], X)$. For $t \in [0, 2\pi]$, we define

$$u(t) = R(t)u(2\pi) + (R * f)(t), \tag{5.34}$$

where

$$u(2\pi) = (I - R(2\pi))^{-1}(R * f)(2\pi). \tag{5.35}$$

If $t = 0$ in (5.34), then $u(0) = u(2\pi)$, because $R(0) = I$. Since $u(0)$ is uniquely determined by (5.35), we obtain that u defined by (5.34) is the unique 2π-periodic mild solution to equation (3.7).

Conversely, assume that for any $f \in C([0, 2\pi], X)$ the equation (3.7) admits precisely one 2π-periodic solution. We first notice that the operator $K : C([0, 2\pi], X) \to C([0, 2\pi], X)$ which takes any function f into the unique 2π-periodic mild solution u to (3.7), that is $(Kf)(t) = u(t)$ is linear and bounded by the closed graph theorem.

Now, suppose that $(I - R(2\pi))x = 0$ for some $x \in X$. Let $u(t) = R(t)x$. Then $u(0) = u(2\pi)$ and hence u is a 2π-periodic mild solution to (3.7) with $f \equiv 0$. The uniqueness implies $u(t) = 0$ for all $t \in [0, 2\pi]$. Thus, $0 = u(2\pi) = R(2\pi)x = x$. We conclude that $(I - R(2\pi))$ is injective.

Next we prove the surjectivity. Let $y \in X$. We need to find $x \in X$ such that $(I - R(2\pi))x = y$. If fact, we first notice that from Definition 5.18, we have that the Laplace transform of $R(t)$ verifies $\tilde{R}(\lambda) = (\lambda - (1 + \tilde{a}(\lambda))A)^{-1}$. Therefore,

$$\frac{d\tilde{R}(\lambda)}{d\lambda} = -\tilde{R}(\lambda)\left[\tilde{R}(\lambda) - \tilde{a}(\lambda)' A\tilde{R}(\lambda)\right], \tag{5.36}$$

for all $\lambda > \omega$. If $c(t) := -ta(t)$, then $\tilde{c}(\lambda) = \frac{d\tilde{a}(\lambda)}{d\lambda} = \tilde{a}(\lambda)'$. The equation (5.36) implies

$$tR(t) = (R * R)(t) - (R * c * AR)(t)$$

for all $t \geq 0$. If we define $H(t) := R(t) - (c * AR)(t)$ then $tR(t) = (R * H)(t)$. Now, we consider the function $h \in C([0, 2\pi], X)$ defined by $h(t) = H(t)y$. Then $tR(t)y = (R * h)(t)$. By hypothesis, there exists a unique 2π-periodic mild solution v when $f = h$. Thus $(Kh)(t) = v(t)$. Let $z := \frac{1}{2\pi}(Kh)(0) = \frac{1}{2\pi}v(0)$. By (5.35) we have

$$\begin{aligned} (I - R(2\pi))(z + y) &= (I - R(2\pi))z + y - R(2\pi)y \\ &= \frac{1}{2\pi}(R * h)(2\pi) + y - R(2\pi)y \\ &= R(2\pi)y + y - R(2\pi)y \\ &= y. \end{aligned}$$

Therefore, $(I - R(2\pi))$ is surjective. Since A is a closed operator, we conclude that $1 \in \rho(R(2\pi))$.

□

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