WELL-POSEDNESS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH MEMORY

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ABSTRACT. In this paper we give characterizations of the existence and uniqueness of Hölder continuous solutions of certain abstract integro-differential equation with memory in terms of a resolvent operator. Moreover, we give necessary conditions in order to ensure the existence and uniqueness of mild solutions on the real line.

1. INTRODUCTION

Let $u(x,t)$ be the temperature of certain material of the point $x \in \Omega$ at the time $t \in \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ $(n = 1,2,3)$ is a bounded open set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The temperature $u(x,t)$ in homogeneous and isotropic media satisfies

$$u_t(x,t) = \kappa \Delta u(x,t), \tag{1.1}$$

where Δ is the Laplacian and $\kappa > 0$ is a constant, called the coefficient of thermal diffusion. This equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory, because in the equation (1.1) the thermal disturbance at any point in the media is felt instantly at every other point. The heat conduction in this kind of materials with fading memory was firstly discussed by Coleman and Gurtin [13], Gurtin and Pipkin [15], and Nunziato [19] among others. In [15] the authors arrived to the heat equation with memory

$$cu_{tt}(x,t) + \alpha(0)u_t(x,t) + \int_{-\infty}^{t} \alpha'(t-s)u_t(x,s)ds = \beta(0)\Delta u(x,t) + \int_{-\infty}^{t} \beta'(t-s)\Delta u(x,s)ds + F(x,t),$$

where $\alpha(t)$ and $\beta(t)$ are positive functions, $c \neq 0$ is a constant called the heat capacity and F is a suitable function. The function a is called the heat-flux relaxation, whereas the function b is known as the energy relaxation function, see for instance [15] for more details. We notice that typical choices of functions α and β are

$$\alpha(t) = \sum_{j=1}^{m} \alpha_i e^{-p_i t}, \quad \beta(t) = \sum_{j=1}^{M} \beta_i e^{-q_i t},$$

where $\alpha_i, \beta_i, p_i, q_i > 0$. We observe that if $\lambda = \frac{\alpha(0)}{c}$, $A = \frac{1}{c}(\alpha'(0)I - \beta(0)\Delta)$, $a(t) = \frac{\beta(0)^{-1}}{c}\alpha'(t)$, $b(t) = \frac{1}{c}[\alpha''(t) - \beta^{-1}(0)\alpha'(0)\beta'(t)]$ and $f(t) = F(\cdot,t)$, then this equation can be written in the abstract form

$$u''(t) + \lambda u'(t) + Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + \int_{-\infty}^{t} b(t-s)u(s)ds = f(t) \quad t \in \mathbb{R}. \tag{1.2}$$

We remark that second order integro-differential equations arise in many fields of applied mathematics, for example in the heat conduction in materials with fading memory, in the description of one-dimensional longitudinal motions of a viscoelastic bar, among others, see for instance [26].

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In this paper, we characterize the well-posedness in Hölder spaces of the second order integro-differential equation with memory

$$u''(t)+\lambda u'(t)+Au(t)+\int_{-\infty}^{t}a(t-s)Au(s)ds+\int_{-\infty}^{t}b(t-s)Bu(s)ds=f(t),\quad t\in\mathbb{R}, \tag{1.3}$$

where $\lambda\in\mathbb{R}$, $A:D(A)\subset X\to X$ and $B:D(B)\subset X\to X$ are closed linear operators defined in a Banach space $X\equiv(X,\|\cdot\|_X)$, the functions $a,b\in L^1(\mathbb{R}_+)$ are suitable kernels and the function f belongs to the Hölder space $C^\alpha(\mathbb{R};X)$. We also present necessary conditions for the existence and uniqueness of mild solutions for equation (1.3). By *well-posedness* of equation (1.3) we understand that for all $f\in C^\alpha(\mathbb{R};X)$ there exists a unique (classical) solution $u\in C^\alpha(\mathbb{R};X)$ for (1.3). We remark that the well-posedness of differential equations is an important tool, because it allows the treatment of semilinear problems. To achieve this, we use some results on vector-valued Fourier multipliers in the Hölder space $C^\alpha(\mathbb{R};X)$ (see [3]). We remark that based on results in [5] and [3] the existence and uniqueness of Hölder type solutions to second order differential equation have been considered by several authors. For example, for the existence and uniqueness of Hölder periodic solutions we refer to [8, 9, 10, 11, 12, 17, 20] and for the existence and uniqueness of Hölder continuous in the real line we mention to [7, 18, 21, 23].

On the other hand, similar methods have been used by several authors to give necessary conditions for the existence and uniqueness of periodic mild solutions of second order differential equations in Banach spaces, see for instance [5, 6, 18, 25]. In the case of mild solution of second order differential equations on the real line, we refer to [25, 27] and the references therein. However, to the best of our knowledge, this problem has not been considered in the case of integro-differential equations in the form of (1.3).

In this paper we are able to give necessary and sufficient conditions in order to obtain the well-poseedness of equation (1.3) in the Hölder space $C^\alpha(\mathbb{R};X)$ $(0<\alpha<1)$ in terms of the *resolvent operator*

$$N_\eta:=\big((i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B\big)^{-1},\quad \eta\in\mathbb{R}.$$

Moreover, we introduce a concept of mild solution for (1.3) and we give necessary condition for the existence and uniqueness of mild solutions to equation (1.3) in terms of the same resolvent operator N_η.

The paper is organized as follows. In Section 2, we review some results about vector-valued Fourier multipliers in the Hölder space $C^\alpha(\mathbb{R};X)$. In Section 3, under suitable conditions on the kernels a and b, we give a characterization of the well-posedness (or maximal regularity) of equation (1.3). In Section 4 we introduce a concept of mild solution to (1.3) and we give a necessary condition for existence and uniqueness of such solutions. Finally, some examples are examined in Section 5.

2. Preliminaries

For Banach spaces X and Y, $\mathcal{B}(X,Y)$ denotes the space of all bounded linear operators from X to Y. If $X=Y$, we write simply $\mathcal{B}(X)$. Now, let $0<\alpha<1$ be fixed. We denote by $C^\alpha(\mathbb{R};X)$ the space of all X-valued functions f on $\mathbb{R}$, such that

$$\|f\|_\alpha=\sup_{t\neq s}\frac{\|f(t)-f(s)\|}{|t-s|^\alpha}<\infty.$$

If we define $\|f\|_{C^\alpha}:=\|f\|_\alpha+\|f(0)\|$, then $(C^\alpha(\mathbb{R};X),\|\cdot\|_{C^\alpha})$ is a Banach space. The kernel of the seminorm $\|\cdot\|_\alpha$ on $C^\alpha(\mathbb{R};X)$ is the space of all constant functions and the corresponding quotient space $\dot{C}^\alpha(\mathbb{R};X)$ is a Banach space in the induced norm. We identify a function $f\in C^\alpha(\mathbb{R};X)$ with its equivalence class

$$\dot{f}:=\{g\in C^\alpha(\mathbb{R};X): f-g\equiv \text{ constant}\}.$$

In this way, $\dot{C}^\alpha(\mathbb{R};X)$ may be identified with the space of all $f\in C^\alpha(\mathbb{R};X)$ such that $f(0)=0$. See [3, Section 5].

We also consider in this paper, the Banach space $C^{\alpha+1}(\mathbb{R};X)$, which consists of all $u \in C^1(\mathbb{R};X)$ such that $u' \in C^\alpha(\mathbb{R};X)$ with the norm

$$\|u\|_{C^{\alpha+1}} = \|u'\|_{C^\alpha} + \|u(0)\|.$$

Analogously, $C^{\alpha+2}(\mathbb{R};X)$ denotes the space of all $u \in C^2(\mathbb{R};X)$ such that $u'' \in C^\alpha(\mathbb{R};X)$. In this case, the norm is defined by

$$\|u\|_{C^{\alpha+2}} = \|u''\|_{C^\alpha} + \|u'(0)\| + \|u(0)\|.$$

Now, we denote by $\mathcal{F}f$, the Fourier transform of f, that is

$$(\mathcal{F}f)(s) := \tilde{f}(s) := \int_{\mathbb{R}} e^{-ist} f(t)dt,$$

for $s \in \mathbb{R}$ and $f \in L^1(\mathbb{R};X)$.

The *Carleman transform of* a function f, denoted by the symbol $\hat{f}(\lambda)$, is defined by

$$\hat{f}(\lambda) = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} f(t), \ \mathrm{Re}\lambda > 0, \\ \displaystyle -\int_{-\infty}^0 e^{-\lambda t} f(t), \ \mathrm{Re}\lambda < 0, \end{cases}$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R};X)$ is of *subexponential growth*, which means

$$\int_{-\infty}^\infty e^{-\epsilon|t|}\|f(t)\|dt < \infty, \quad \text{for each } \epsilon > 0.$$

The Laplace transform of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$ is denoted by

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \mathrm{Re}\lambda > \omega,$$

whenever the integral is absolutely convergent for $\mathrm{Re}\lambda > \omega$. Observe that we use the same symbol for the Carleman and Laplace transform but, this will not lead to confusion.

The relation between the Laplace transform of $f \in L^1(\mathbb{R};X)$, $f(t) = 0$ for $t < 0$, and its Fourier transform is

$$\mathcal{F}(f)(s) = \hat{f}(is), \quad s \in \mathbb{R}.$$

When $f \in L^1(\mathbb{R};X)$ is of subexponential growth, we have by [4, Chapter 4],

$$\lim_{\sigma\to 0^+} (\hat{f}(\sigma + i\rho) - \hat{f}(-\sigma + i\rho)) = \tilde{f}(\rho), \quad \rho \in \mathbb{R}. \tag{2.1}$$

If $a \in L^1(\mathbb{R}_+)$, we will always identify a with its extension on $\mathbb{R}$ by letting $a(t) = 0$ for $t < 0$. In such way, when $a \in L^1(\mathbb{R}_+)$, the Fourier transform $\tilde{a}(\rho)$ makes sense for all $\rho \in \mathbb{R}$. Moreover, by (2.1) we have

$$\lim_{\sigma\to 0^+} \hat{a}(\sigma + i\rho) = \tilde{a}(\rho)$$

and $\hat{a}(-\sigma + i\rho) = 0$ for all $\sigma > 0$ and $\rho \in \mathbb{R}$ by definition.

In what follows, we always assume that $\tilde{a}(\eta) \neq -1$, for all $\eta \in \mathbb{R}$, and we use the following notation:

$$a_\eta := \tilde{a}(\eta), \quad \eta \in \mathbb{R}.$$

Now, we recall the notion of regular kernels (see [26, p. 69]).

Definition 2.1. *Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. The kernel $a(t)$ is called* k-regular *if there is a constant $c > 0$ such that*

$$|\lambda^n[\hat{a}(\lambda)]^{(n)}| \leq c|\hat{a}(\lambda)|, \quad \textit{for all} \quad \mathrm{Re}(\lambda) > 0, 0 \leq n \leq k.$$

For the reader's convenience, we summarize here from [26, Lemma 8.1] some properties of 1-regular kernels.

Lemma 2.2. *Suppose that $b \in L^1_{\text{loc}}(\mathbb{R}_+)$ is of sub-exponential growth and* 1*-regular. Then*
(*i*) $\hat{b}(i\rho) := \lim_{\lambda \to i\rho} \hat{b}(\lambda)$ *exists for each* $\rho \neq 0$;
(*ii*) $\hat{b}(\lambda) \neq 0$ *for each* $Re(\lambda) \geq 0$, $\lambda \neq 0$;
(*iii*) $\hat{b}(i\cdot) \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \setminus \{0\})$;
(*iv*) $|\rho[\hat{b}(i\rho)]'| \leq c|\hat{b}(i\rho)|$ *for a.a.* $\rho \in \mathbb{R}$.

We denote by $L^1(\mathbb{R}_+, t^\alpha dt)$ the set of all $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ such that

$$\int_0^\infty |a(t)|t^\alpha dt < \infty. \tag{2.2}$$

Observe that as consequence such a is always in $L^1(\mathbb{R}_+)$. Given $v \in C^\alpha(\mathbb{R}; X)$ $(0 < \alpha < 1)$ and $a \in L^1(\mathbb{R}_+, t^\alpha dt)$, we write

$$(a \dot{*} v)(t) := \int_{-\infty}^t a(t-s)v(s)ds = \int_0^\infty a(s)v(t-s)ds. \tag{2.3}$$

From (2.2) the above integral is well defined. Moreover, it follows from the definition that

$$\text{if } v \in C^\alpha(\mathbb{R}; X) \text{ then } a \dot{*} v \in C^\alpha(\mathbb{R}; X) \text{ and } \|a \dot{*} v\|_\alpha \leq \|a\|_1 \|v\|_\alpha. \tag{2.4}$$

Observe that with this notation, the Equation (1.3) can be written as

$$u''(t) + \lambda u'(t) + Au(t) + (a \dot{*} Au)(t) + (b \dot{*} Bu)(t) = f(t), \quad t \in \mathbb{R}.$$

Let Ω be an open set in $\mathbb{R}$. By $C_c^\infty(\Omega)$ we denote the space of all $C^\infty-$functions in Ω having compact support in Ω.

Definition 2.3. *Let $N : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X, Y)$ be continuous. We say that N is a $\dot{C}^\alpha-$multiplier if there exists a map $L : \dot{C}^\alpha(\mathbb{R}; X) \to \dot{C}^\alpha(\mathbb{R}; Y)$ such that*

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot N))(s)f(s)ds \tag{2.5}$$

for all $f \in C^\alpha(\mathbb{R}; X)$ and all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Here $(\mathcal{F}(\phi \cdot N))(s) = \int_{\mathbb{R}} e^{-ist}\phi(t)N(t)dt \in \mathcal{B}(X, Y)$. Observe that the right-hand side of (2.5) does not depend on the representative of $\dot{f}$ because

$$\int_{\mathbb{R}} (\mathcal{F}(\phi N)(s))(s)ds = 2\pi(\phi N)(0) = 0.$$

Therefore, if L exists, then it is well defined. Moreover, left-hand side of (2.5) determines the function $Lf \in C^\alpha(\mathbb{R}; X)$ uniquely up to some constant (by [3, Lemma 5.1]). Moreover, if (2.5) holds, then $L : \dot{C}^\alpha(\mathbb{R}; X) \to \dot{C}^\alpha(\mathbb{R}; Y)$ is linear and continuous (see [3, Definition 5.2]) and if $f \in C^\alpha(\mathbb{R}; X)$ is bounded, then Lf is bounded as well (see [3, Remark 6.3]).

The following multiplier theorem is due to Arendt, Batty and Bu.

Theorem 2.4. [3, Theorem 5.3] *Let $N \in C^2(\mathbb{R} \setminus \{0\}; \mathcal{B}(X, Y))$ be such that*

$$\sup_{t \neq 0} \|N(t)\| + \sup_{t \neq 0} \|tN'(t)\| + \sup_{t \neq 0} \|t^2 N''(t)\| < \infty. \tag{2.6}$$

Then, N is a $\dot{C}^\alpha-$multiplier.

Example 2.5. *Let X be a Banach space and $0 < \alpha < 1$. Define $N(t) = I$ for $t \geq 0$ and $N(t) = 0$ for $t < 0$. It follows from Theorem 2.4 that N is a $\dot{C}^\alpha$-multiplier. The associated operator on $\dot{C}^\alpha(\mathbb{R}; X)$ is called the* Riesz projection.

Example 2.6. *Let X be a Banach space and $0<\alpha<1$. Define $N(t)=(-i\,\mathrm{sign}t)I$ for $t\in\mathbb{R}$. Then N is a $\dot{C}^\alpha$-multiplier by Theorem 2.4. The associated operator on $\dot{C}^\alpha(\mathbb{R};X)$ is called the* Hilbert transform.

Recall that a Banach space X has the *Fourier type* p, with $1\le p\le 2$, if the Fourier transform defines a bounded linear operator from $L^p(\mathbb{R};X)$ to $L^q(\mathbb{R};X)$, where $1/p+1/q=1$. We notice that the space $L^p(\Omega)$ with $1\le p\le 2$ has Fourier type p; a Banach space X has the Fourier type 2 if and only if X is isomorphic to a Hilbert space; X has Fourier type p if and only if X^* has Fourier type p. Every Banach space has Fourier type 1. A Banach space X is said to be $B-$convex if it has Fourier type p, for some $p>1$. Every uniformly convex space is $B-$convex. For more details of B-convex spaces, see for instance [16].

Remark 2.7.

If X is $B-$convex, in particular if X is a UMD space, then the Theorem 2.4 holds if the condition (2.6) is replaced by the weaker condition

$$\sup_{t\neq 0}\|N(t)\|+\sup_{t\neq 0}\|tN'(t)\|<\infty, \tag{2.7}$$

where $N\in C^1(\mathbb{R}\setminus\{0\};\mathcal{B}(X,Y))$, see [3, Remark 5.5].

Now, we recall the following results.

Lemma 2.8. [3] *Let $f\in C^\alpha(\mathbb{R};X)$. Then f is constant if and only if $\int_{\mathbb{R}}f(s)(\mathcal{F}\varphi)(s)ds=0$ for all $\varphi\in C_c^\infty(\mathbb{R}\setminus\{0\})$.*

Define $\mathrm{id}:\mathbb{R}\to\mathbb{C}$ by $\mathrm{id}(s)=is$.

Lemma 2.9. [3] *Let $0<\alpha<1$, $u,v\in C^\alpha(\mathbb{R};X)$. Then, the following assertions are equivalent,*

(i) $u\in C^{\alpha+1}(\mathbb{R};X)$ and $u'-v$ is constant;

(ii) $\int_{\mathbb{R}}v(s)\mathcal{F}(\phi)(s)ds=\int_{\mathbb{R}}u(s)\mathcal{F}(\mathrm{id}\cdot\phi)(s)ds$, for all $\phi\in C_c^\infty(\mathbb{R}\setminus\{0\})$.

Lemma 2.10. *Let $0<\alpha<1$, $u,v\in C^\alpha(\mathbb{R};X)$. Then, the following assertions are equivalent,*

(i) $u\in C^{\alpha+2}(\mathbb{R};X)$ and $u''-v$ is constant;

(ii) $\int_{\mathbb{R}}v(s)\mathcal{F}(\phi)(s)ds=\int_{\mathbb{R}}u(s)\mathcal{F}(\mathrm{id}^2\cdot\phi)(s)ds$, for all $\phi\in C_c^\infty(\mathbb{R}\setminus\{0\})$.

The following Lemma, is a direct consequence of [18, Lemma 3.2].

Lemma 2.11. *Let $0<\alpha<1$, $v\in C^\alpha(\mathbb{R};[D(A)])$, $u\in C^\alpha(\mathbb{R};X)$ and $a\in L^1(\mathbb{R}_+,t^\alpha dt)$. The following assertions are equivalent,*

(i) $a{*}Av-u$ is constant;

(ii) $\int_{\mathbb{R}}u(s)(\mathcal{F}\phi)(s)ds=\int_{\mathbb{R}}Av(s)\mathcal{F}(a_s\phi)(s)ds$, for all $\phi\in C_c^\infty(\mathbb{R}\setminus\{0\})$.

Let $f\in L^1(\mathbb{R},(1+|t|)^{-k}dt;X)$, where $k\in\mathbb{N}_0$. We define $\mathcal{F}f$ as a linear mapping from $C_c^\infty(\mathbb{R}\setminus\{0\})$ into X by

$$\langle\varphi,\mathcal{F}f\rangle=\int_{\mathbb{R}}f(t)(\mathcal{F}\varphi)(t)dt,\quad \varphi\in C_c^\infty(\mathbb{R}\setminus\{0\}).$$

The next lemma follows from [4, Theorems 4.8.1 and 4.8.2].

Lemma 2.12. *Let $f\in L^1(\mathbb{R},(1+|t|)^{-k}dt;X)$, where $k\in\mathbb{N}_0$. Then f is constant if and only if $\langle\varphi,\mathcal{F}f\rangle=\int_{\mathbb{R}}f(s)(\mathcal{F}\varphi)(s)ds=0$ for all $\varphi\in C_c^\infty(\mathbb{R}\setminus\{0\})$.*

3. C^α-well posedness

In this section we study the well-posedness of equation (1.3) in the Hölder space $C_B^\alpha(\mathbb{R};X)$. Given a kernel a and a closed operator A we define the space

$$C_A^{\alpha,a}(\mathbb{R};X):=\{v\in C^\alpha(\mathbb{R};[D(A)]):\exists w\in C^\alpha(\mathbb{R};X)\text{ such that }w-(a{*}Av)\text{ is constant}\}.$$

Now, we define the following solution space:

$$\mathcal{S}:=C^{\alpha+2}(\mathbb{R};X)\cap C^\alpha(\mathbb{R};[D(A)])\cap C_A^{\alpha,a}(\mathbb{R};X)\cap C_B^{\alpha,b}(\mathbb{R};X).$$

Definition 3.1. *We say that the equation (1.3) is* C^α-well posed *if for each* $f \in C^\alpha(\mathbb{R}; X)$, *there exists a unique function* $u \in \mathcal{S}$ *such that the equation (1.3) holds for all* $t \in \mathbb{R}$.

Remark 3.2.

We notice that if (1.3) is C^α-well posed, then it follows from the closed graph theorem that the map $L : C^\alpha(\mathbb{R}; X) \to \mathcal{S}$, which associates to the function f the unique solution u of (1.3) is linear and continuous. Indeed, since A and B are linear closed operators, the space $\mathcal{S}$ endowed with the norm

$$\|u\|_H := \|u''\|_{C^\alpha} + |\lambda|\,\|u'\|_{C^\alpha} + \|Au\|_{C^\alpha} + \|(a\dot{*}Au)\|_{C^\alpha} + \|(b\dot{*}Bu)\|_{C^\alpha}$$

is a Banach space.

For $a, b \in L^1_{\text{loc}}(\mathbb{R}_+)$ we define the *resolvent set* $\rho_{a,b}(A, B)$ as

$$\rho_{a,b}(A,B) = \{\mu \in \mathbb{C} : (\mu^2 + \lambda\mu + (1 + \hat{a}(\mu))A + \hat{b}(\mu)B) : D(A) \cap D(B) \to X$$
$$\text{is invertible and } (\mu^2 + \lambda\mu + (1 + \hat{a}(\mu))A + \hat{b}(\mu)B)^{-1} \in \mathcal{B}(X)\},$$

where $\hat{a}(\cdot)$ and $\hat{b}(\cdot)$ denote the Laplace transform of a and b respectively.

Proposition 3.3. *Let* $a, b \in L^1(\mathbb{R}_+, t^\alpha dt)$. *Let* $A : D(A) \subset X \to X$ *and* $B : D(B) \subset X \to X$ *be closed linear operators defined in a Banach space* X *with* $D(A) \cap D(B) \neq \{0\}$. *For* $\eta \in \mathbb{R}$ *we write* $N_\eta := \big((i\eta)^2 + \lambda(i\eta) + (1 + a_\eta)A + b_\eta B\big)^{-1}$. *If the problem (1.3) is* C^α*-well posed, then*

(*i*) $i\eta \in \rho_{a,b}(A, B)$ *for all* $\eta \in \mathbb{R}$, *and;*

(*ii*)
$$\sup_{\eta\in\mathbb{R}} \|\eta^2 N_\eta\| < \infty, \quad \sup_{\eta\in\mathbb{R}} \|a_\eta A N_\eta\| < \infty \quad \textit{and} \quad \sup_{\eta\in\mathbb{R}} \|b_\eta B N_\eta\| < \infty.$$

Proof. Let $\eta \in \mathbb{R}$ and suppose that

$$[(i\eta)^2 + \lambda(i\eta) + (1 + a_\eta)A + b_\eta B]x = 0 \tag{3.1}$$

where $x \in D(A) \cap D(B)$. Let $u(t) = e^{i\eta t}x$. Then, u is a solution to (1.3) with $f \equiv 0$. In fact, since

$$(a\dot{*}Au)(t) = \int_{-\infty}^t a(t-s)Ae^{i\eta s}x ds = e^{i\eta t}\int_0^\infty a(v)e^{-i\eta v}Ax dv = e^{i\eta t}a_\eta Ax,$$

by (3.1) we have

$$u''(t) + \lambda u'(t) + Au(t) + (a\dot{*}Au)(t) + (b\dot{*}Bu)(t) = e^{i\eta t}[(i\eta)^2 + \lambda(i\eta) + (1 + a_\eta)A + b_\eta B]x = 0.$$

From the uniqueness it follows that $u \equiv 0$, which implies that $x = 0$. Therefore, $[(i\eta)^2 + \lambda(i\eta) + (1 + a_\eta)A + b_\eta B]$ is injective.

Now, we shall prove the surjectivity. Let $y \in X$. Let $L : C^\alpha(\mathbb{R}; X) \to \mathcal{S}$ be the bounded operator which takes each $f \in C^\alpha(\mathbb{R}; X)$ to the unique solution u of equation (1.3). Let $\eta \in \mathbb{R}$, $f(t) = e^{i\eta t}y$ and $u = Lf$. Note that for fixed $s \in \mathbb{R}$ we have that $v_1(t) := u(t + s)$ and $v_2(t) := e^{i\eta s}u(t)$ are both solutions of (1.3) with $g(t) = e^{is\eta}f(t)$. By uniqueness $v_1 = v_2$, that is, $u(t + s) = e^{is\eta}u(t)$ for all $s, t \in \mathbb{R}$. Let $x = u(0) \in D(A) \cap D(B)$. Then, $u(t) = e^{i\eta t}x$ and u satisfies the equation (1.3). Now, observe that

$$(a\dot{*}Au)(t) = e^{i\eta t}a_\eta Ax \quad \text{and} \quad (b\dot{*}Bu)(t) = e^{i\eta t}b_\eta Bx \quad \text{for all} \quad t \in \mathbb{R}.$$

In particular, $(a\dot{*}Au)(0) = a_\eta Ax$ and $(b\dot{*}Bu)(0) = b_\eta Bx$. Since $u'(t) = (i\eta)e^{i\eta t}x$ and $u''(t) = (i\eta)^2e^{i\eta t}x$ we obtain $u'(0) = (i\eta)x$ and $u''(0) = (i\eta)^2x$ and thus,

$$[(i\eta)^2 + \lambda(i\eta) + (1 + a_\eta)A + b_\eta B]x = u''(0) + \lambda u'(0) + Au(0) + (a\dot{*}Au)(0) + (b\dot{*}Bu)(0) = f(0) = y. \tag{3.2}$$

We conclude that the operator $[(i\eta)^2 + \lambda(i\eta) + (1 + a_\eta)A + b_\eta B]$ is surjective and therefore $[(i\eta)^2 + \lambda(i\eta) + (1 + a_\eta)A + b_\eta B]$ is invertible.

On the other hand, by (3.2) we obtain $x = N_\eta y$ and therefore

$$\|N_\eta y\| = \|x\| = \|Lf(0)\| \leq \|L\|\,\|f(0)\| = \|L\|\,\|y\|.$$

Since $y \in X$ is arbitrary, we obtain that $[(i\eta)^2 + \lambda(i\eta) + (1+a_\eta)A + b_\eta B]^{-1}$ is a bounded operator for all $\eta \in \mathbb{R}$, which means that $\{i\eta\}_{\eta\in\mathbb{R}} \subset \rho_{a,b}(A,B)$.

Let $y \in X$. We first notice that for $f(t) = e^{i\eta t}y$ the solution u to (1.3) is given by $u(t) = e^{i\eta t}x$, and therefore $u(t) = e^{i\eta t}N_\eta y$. Denote $e_\eta \otimes x$ to the function $t \mapsto (e_\eta \otimes x)(t) := e^{i\eta t}x$. Since $\|e_\eta \otimes x\|_\alpha = \gamma_\alpha |\eta|^\alpha \|x\|$, where $\gamma_\alpha = 2\sup_{t>0} t^{-\alpha}\sin(t/2)$ (see [3, Section 3]) we have

$$\begin{aligned}
\gamma_\alpha |\eta|^\alpha \|(i\eta)^2 N_\eta y\| &= \|e_\eta \otimes (i\eta)^2 N_\eta y\|_\alpha \\
&= \|u''\|_\alpha \le \|u''\|_{C^\alpha} \le \|u\|_H \\
&= \|Lf\|_H \le \|L\|\,\|f\|_{C^\alpha} \le \|L\|(\|f\|_\alpha + \|f(0)\|) \\
&= \|L\|(\gamma_\alpha|\eta|^\alpha + 1)\|y\|.
\end{aligned}$$

Therefore, $\|(i\eta)^2 N_\eta\| \le \|L\|(1 + \gamma_\alpha^{-1}|\eta|^{-\alpha})$ and thus

$$\sup_{|\eta|\ge 1} \|(i\eta)^2 N_\eta\| < \infty.$$

Since the function $\eta \mapsto (i\eta)^2 N_\eta$ is continuous in $\mathbb{R}$, it follows from the compactness that

$$\sup_{|\eta|\le 1} \|(i\eta)^2 N_\eta\| < \infty.$$

Therefore,

$$\sup_{\eta\in\mathbb{R}} \|\eta^2 N_\eta\| < \infty.$$

On the other hand, since $(b \dot{*} Bu)(t) = e^{i\eta t} b_\eta Bx = e^{i\eta t} b_\eta B N_\eta y$ we have

$$\begin{aligned}
\gamma_\alpha |\eta|^\alpha \|b_\eta B N_\eta y\| &= \|e_\eta \otimes b_\eta B N_\eta y\|_\alpha \\
&= \|(b \dot{*} Bu)\|_\alpha \le \|(b \dot{*} Bu)\|_{C^\alpha} \le \|u\|_H \\
&= \|Lf\|_H \le \|L\|\,\|f\|_{C^\alpha} \le \|L\|(\|f\|_\alpha + \|f(0)\|) \\
&= \|L\|(\gamma_\alpha|\eta|^\alpha + 1)\|y\|.
\end{aligned}$$

We conclude analogously to the proof of $\sup_{\eta\in\mathbb{R}} \|\eta^2 N_\eta\| < \infty$ that $\sup_{\eta\in\mathbb{R}} \|b_\eta B N_\eta\| < \infty$. A similar computation shows that $\sup_{\eta\in\mathbb{R}} \|a_\eta A N_\eta\| < \infty$.

□

The following Theorem is one of the main result in this paper, which shows that under an additional hypothesis (the 2-regularity of kernel a) we can prove the converse of Proposition 3.3.

Theorem 3.4. *Let $a, b \in L^1(\mathbb{R}_+, t^\alpha dt)$ be 2-regular kernels. Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed linear operators defined in a Banach space X with $D(A) \cap D(B) \ne \{0\}$. For $\eta \in \mathbb{R}$ we write $N_\eta := \big((i\eta)^2 + \lambda(i\eta) + (1+a_\eta)A + b_\eta B\big)^{-1}$. Then, the following assertions are equivalent*

(i) The equation (1.3) is C^α-well posed;

(ii) $\{i\eta\}_{\eta\in\mathbb{R}} \subset \rho_{a,b}(A,B)$ and

$$\sup_{\eta\in\mathbb{R}} \|\eta^2 N_\eta\| < \infty, \quad \sup_{\eta\in\mathbb{R}} \|a_\eta A N_\eta\| < \infty \quad \textit{and} \quad \sup_{\eta\in\mathbb{R}} \|b_\eta B N_\eta\| < \infty.$$

Proof. $(i) \Rightarrow (ii)$. It follows from Proposition 3.3.

$(ii) \Rightarrow (i)$. For $t \in \mathbb{R}$, we define the operators $N(t) := \big((it)^2 + \lambda(it) + (1+a_t)A + b_t B\big)^{-1}$ and $M(t) := (it)^2 N(t)$. Observe that by hypothesis $N \in C^2(\mathbb{R}; \mathcal{B}(X, [D(A) \cap D(B)]))$. We claim that N is a $\dot{C}^\alpha$-multiplier. In fact, the identity

$$\big((it)^2 + i\lambda t + (1+a_t)A + b_t B\big) N(t) = I \tag{3.3}$$

implies that

$$(1+a_t)AN(t) = -(it)^2 N(t) - i\lambda t N(t) - b_t B N(t), \tag{3.4}$$

for all $t \in \mathbb{R}$. Note that if $|t| \geq 1$ then $(1+a_t)AN(t) = -M(t) + \frac{i\lambda}{t}M(t) - b_tBN(t)$ and thus by hypothesis $\sup_{|t|\geq 1} \| -M(t) + \frac{i\lambda}{t}M(t) - b_tBN(t)\| < \infty$. Since $t \mapsto N(t)$ is continuous, the compactness of $[-1,1]$ and the hypothesis imply that $\sup_{|t|\leq 1} \| -(it)^2N(t) - i\lambda tN(t) - b_tBN(t)\| < \infty$. Hence $\sup_{t\in\mathbb{R}} \|(1+a_t)AN(t)\| < \infty$.

On the other hand, the identity (3.3) implies $(it)^2N(t) = I - i\lambda tN(t) - (1+a_t)AN(t) - b_tBN(t)$ and thus $\sup_{|t|\geq 1} \|N(t)\| < \infty$. Finally, we conclude by the continuity of $N(t)$ on $[-1,1]$ that $\sup_{|t|\leq 1} \|N(t)\| < \infty$. Therefore, $\sup_{t\in\mathbb{R}} \|N(t)\| < \infty$. A similar argument shows that $\sup_{t\in\mathbb{R}} \|tN(t)\| < \infty$. Moreover, from the identity (3.4) we obtain $\sup_{t\in\mathbb{R}} \|AN(t)\| < \infty$.

Now, an easy computation shows that,

$$N'(t) = -N(t)[2i(it) + i\lambda + a_t'A + b_t'B]N(t), \tag{3.5}$$

and

$$N''(t) = 2N(t)[-2t + i\lambda + a_t'A + b_t'B]N(t)[-2t + i\lambda + a_t'A + b_t'B]N(t) - N(t)[-2 + a_t''A + b_t''B]N(t). \tag{3.6}$$

The 2-regularity of a and b implies

$$\|tN'(t)\| \leq 2\|t^2N(t)\| + |\lambda|\|tN(t)\| + \|a_tAN(t)\| + \|b_tBN(t)\|,$$

$$\begin{aligned} \|t^2N''(t)\| \quad &\leq \quad 2\|N(t)\| \left[2\|t^2N(t)\| + |\lambda|\,\|N(t)\| + \|a_tAN(t)\| + \|b_tBN(t)\|\right]^2 \\ &\quad + \|N(t)\| \left[2\|t^2N(t)\| + \|a_tAN(t)\| + \|b_tBN(t)\|\right], \end{aligned}$$

for all $t \in \mathbb{R}$. From the hypothesis we obtain that

$$\sup_{t\in\mathbb{R}} \|tN'(t)\| < \infty \text{ and } \sup_{t\in\mathbb{R}} \|t^2N''(t)\| < \infty. \tag{3.7}$$

We conclude from Theorem 2.4 that N is a $\dot{C}^\alpha$-multiplier, with $N \in C^2(\mathbb{R}; \mathcal{B}(X, [D(A) \cap D(B)]))$.

Next, we define the operator $P \in C^2(\mathbb{R}; \mathcal{B}(X, [D(A) \cap D(B)]))$ by $P(t) := (\mathrm{id}^2 \cdot N)(t)$. Observe that by hypothesis $\sup_{t\in\mathbb{R}} \|P(t)\| < \infty$. On the other hand,

$$P'(t) = 2i(it)N(t) + (it)^2N'(t),$$

$$P''(t) = -2N(t) + 4i(it)N'(t) + (it)^2N''(t),$$

and

$$tP'(t) = 2(it)^2N(t) + (it)^2tN'(t),$$

$$t^2P''(t) = -2t^2N(t) + 4(it)^2tN'(t) + (it)^2t^2N''(t).$$

The identities (3.5)-(3.6), and (3.7) imply that $\sup_{t\in\mathbb{R}} \|(it)^2tN'(t)\| < \infty$ and $\sup_{t\in\mathbb{R}} \|(it)^2t^2N''(t)\| < \infty$. From hypothesis we conclude $\sup_{t\in\mathbb{R}} \|tP'(t)\| < \infty$ and $\sup_{t\in\mathbb{R}} \|t^2P''(t)\| < \infty$. This implies that P is a $\dot{C}^\alpha$-multiplier by Theorem 2.4. Similar computations show that $Q \in C^2(\mathbb{R}; \mathcal{B}(X, [D(A)]))$, $R \in C^2(\mathbb{R}; \mathcal{B}(X, [D(B)]))$ and $S \in C^2(\mathbb{R}; \mathcal{B}(X, [D(A) \cap D(B)]))$ defined respectively by $Q(t) := (1+a_t)AN(t)$ and $R(t) := b_tBN(t)$ and $S(t) := \lambda tN(t)$ are $\dot{C}^\alpha$-multipliers.

Let $f \in C^\alpha(\mathbb{R}; X)$. Since N, P, Q, R and S are $\dot{C}^\alpha$-multipliers, there exist $\overline{u} \in C^\alpha(\mathbb{R}; [D(A) \cap D(B)])$, $v \in C^\alpha(\mathbb{R}; [D(A) \cap D(B)])$, $w \in C^\alpha(\mathbb{R}; [D(A)])$, $x \in C^\alpha(\mathbb{R}; [D(B)])$ and $y \in C^\alpha(\mathbb{R}; [D(A) \cap D(B)])$ such that

$$\int_{\mathbb{R}} \overline{u}(s)(\mathcal{F}\phi_1)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_1 \cdot N)(s)f(s)ds, \tag{3.8}$$

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\phi_2)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_2 \cdot P)(s)f(s)ds, \tag{3.9}$$

$$\int_{\mathbb{R}} w(s)(\mathcal{F}\phi_3)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_3 \cdot Q)(s)f(s)ds, \tag{3.10}$$

$$\int_{\mathbb{R}} x(s)(\mathcal{F}\phi_4)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_4 \cdot R)(s)f(s)ds, \tag{3.11}$$

$$\int_{\mathbb{R}} y(s)(\mathcal{F}\phi_5)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_5 \cdot S)(s)f(s)ds, \tag{3.12}$$

for all $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Letting $\phi_1 = \mathrm{id}^2 \cdot \phi_2$ in (3.8) we obtain from (3.9)

$$\int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(\mathrm{id}^2 \cdot \phi_2)(s)ds = \int_{\mathbb{R}} v(s)(\mathcal{F}\phi_2)(s)ds. \tag{3.13}$$

By Lemma 2.10 we have $\overline{u} \in C^{\alpha+2}(\mathbb{R}; X)$ and $v(t) = \overline{u}''(t) + y_0$, where $y_0 \in X$.

Observe that $\overline{u}(t) \in D(A) \cap D(B)$ and $\mathcal{F}(\phi_1 \cdot N)(s)x \in D(A) \cap D(B)$ for all $x \in X$, $\phi_1 \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Now, we choose $\phi_1 = (1 + a.) \cdot \phi_3$ in (3.8). Since A is a closed operator we have from (3.10)

$$\int_{\mathbb{R}} A\overline{u}(s)\mathcal{F}((1 + a_s) \cdot \phi_3)(s)ds = \int_{\mathbb{R}} w(s)(\mathcal{F}\phi_3)(s)ds. \tag{3.14}$$

From Lemma 2.11 we obtain $\overline{u} \in C_A^{\alpha,a}(\mathbb{R}; X)$ and $w(t) = A\overline{u}(t) + (a \ast A\overline{u})(t) + y_1$, where $y_1 \in X$. Similarly, if $\phi_1 = b. \cdot \phi_4$ in (3.8) we have from (3.11) (because B is a closed operator) that

$$\int_{\mathbb{R}} B\overline{u}(s)\mathcal{F}(b_s \cdot \phi_4)(s)ds = \int_{\mathbb{R}} x(s)(\mathcal{F}\phi_4)(s)ds, \tag{3.15}$$

which implies $\overline{u} \in C_B^{\alpha,b}(\mathbb{R}; X)$ and $x(t) = (b \ast B\overline{u})(t) + y_2$, where $y_2 \in X$. Finally, if $\phi_1 = \lambda \mathrm{id} \cdot \phi_5$ in (3.8) we have from (3.12) that

$$\int_{\mathbb{R}} \lambda\overline{u}(s)\mathcal{F}(\mathrm{id} \cdot \phi_5)(s)ds = \int_{\mathbb{R}} y(s)(\mathcal{F}\phi_5)(s)ds, \tag{3.16}$$

which implies $y(t) = \lambda\overline{u}'(t) + y_3$, where $y_3 \in X$.

Observe that (3.8)–(3.12) and the identity $(it)^2N(t) = I - i\lambda tN(t) - (1 + a_t)AN(t) - b_tBN(t)$ imply

$$\begin{aligned}
\int_{\mathbb{R}} v(s)(\mathcal{F}\phi_2)(s)ds &= \int_{\mathbb{R}} \mathcal{F}(\mathrm{id}^2 \cdot \phi_2 \cdot N)(s)f(s)ds \\
&= \int_{\mathbb{R}} \mathcal{F}(\phi_2 \cdot [I - \lambda \mathrm{id} \cdot N - (1 + a.)AN - b.BN]f(s)ds \\
&= \int_{\mathbb{R}} \mathcal{F}(\phi_2)(s)f(s)ds - \int_{\mathbb{R}} \mathcal{F}(\phi_2 \cdot S)(s)f(s)ds - \int_{\mathbb{R}} \mathcal{F}(\phi_2 \cdot Q)(s)f(s)ds \\
&\quad - \int_{\mathbb{R}} \mathcal{F}(\phi_2 \cdot R)(s)f(s)ds \\
&= \int_{\mathbb{R}} \mathcal{F}(\phi_2)(s)f(s)ds - \int_{\mathbb{R}} y(s)(\mathcal{F}\phi_2)(s)ds - \int_{\mathbb{R}} w(s)(\mathcal{F}\phi_2)(s)ds \\
&\quad - \int_{\mathbb{R}} x(s)(\mathcal{F}\phi_2)(s)ds.
\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} [v(s) + y(s) + w(s) + x(s)](\mathcal{F}\phi_2)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_2)(s)f(s)ds. \tag{3.17}$$

It follows from (3.17) and Lemma 2.12 that $v(t) + y(t) + w(t) + x(t) = f(t) + y_4$ where $y_4 \in X$. Therefore $\overline{u}''(t) + \lambda\overline{u}'(t) + A\overline{u}(t) + (a \ast A\overline{u})(t) + (b \ast B\overline{u})(t) = f(t) + y$, where $y = y_4 - (y_0 + y_1 + y_2 + y_3)$. Let $u(t) = \overline{u}(t) + x$ where $x = [(1 + a_0)A + b_0B]^{-1}y$. Note that x is well defined because $\{i\eta\}_{\eta \in \mathbb{R}} \subset \rho_{a,b}(A, B)$. We observe that u is a solution of (1.3). In fact, since $(a\ast A\overline{u})(t) = (a\ast Au)(t) + \int_0^\infty a(r)xdr = (a\ast Au)(t) + a_0x$ we obtain

$$\begin{aligned} u''(t) &= \overline{u}''(t) \\ &= f(t) - [\lambda\overline{u}'(t) + A\overline{u}(t) + (a\ast A\overline{u})(t) + (b\ast B\overline{u})(t)] + y \\ &= f(t) - \lambda u'(t) - Au(t) - (a\ast Au)(t) - (b\ast Bu)(t) - (1+a_0)Ax - b_0Bx + y \\ &= f(t) - \lambda u'(t) - Au(t) - (a\ast Au)(t) - (b\ast Bu)(t). \end{aligned}$$

On the other hand, since $\overline{u} \in C^\alpha(\mathbb{R};[D(A)\cap D(B)]) \cap C^{\alpha+2}(\mathbb{R};X) \cap C^{\alpha,a}_A(\mathbb{R};X) \cap C^{\alpha,b}_B(\mathbb{R};X)$ we have $u \in C^\alpha(\mathbb{R};[D(A)]) \cap C^{\alpha,2}(\mathbb{R};X) \cap C^{\alpha,a}_A(\mathbb{R};X) \cap C^{\alpha,b}_B(\mathbb{R};X)$ and therefore, u is a solution of equation (1.3).

In order to prove uniqueness, suppose that

$$u''(t) + \lambda u'(t) + Au(t) + (a\ast Au)(t) + (b\ast Bu)(t) = 0, \quad t \in \mathbb{R}. \tag{3.18}$$

As in [18, Appendix A], for $\sigma > 0$, we denote $L_\sigma(u)(\rho)$ by $L_\sigma(u)(\rho) := \hat{u}(\sigma + i\rho) - \hat{u}(-\sigma + i\rho)$, where $\rho \in \mathbb{R}$. Take L_σ in (3.18). From [18, Proposition A.2.(iv)], we have

$$L_\sigma(u'')(\rho) + \lambda L_\sigma(u')(\rho) + AL_\sigma(u)(\rho) + A\hat{a}(\sigma+i\rho)L_\sigma(u)(\rho) + G^{Au}_a(\sigma,\rho) + B\hat{b}(\sigma+i\rho)L_\sigma(u)(\rho) + G^{Bu}_b(\sigma,\rho) = 0, \tag{3.19}$$

with

$$\lim_{\sigma\to 0^+} \int_{\mathbb{R}} G^{Au}_a(\sigma,\rho)\phi(\rho)d\rho = \lim_{\sigma\to 0^+} \int_{\mathbb{R}} G^{Bu}_b(\sigma,\rho)\phi(\rho)d\rho = 0,$$

for all $\phi \in \mathcal{S}(\mathbb{R})$, where

$$\begin{aligned} G^{Au}_a(\sigma,\rho) &= \int_{-\infty}^0 \left(\int_{-s}^\infty a(\tau)e^{-(\sigma+i\rho)\tau}d\tau\right) e^{-(\sigma+i\rho)s}Au(s)ds \\ &+ \int_{-\infty}^0 \left(\int_0^{-s} a(\tau)e^{(\sigma-i\rho)(s+\tau)}d\tau\right) Au(s)ds \\ &- \int_{-\infty}^0 \left(\int_0^\infty a(\tau)e^{-(\sigma+i\rho)\tau}d\tau\right) e^{(\sigma-i\rho)s}Au(s)ds, \end{aligned}$$

and $G^{Bu}_b(\sigma,\rho)$ is defined analogously. By [18, Proposition A.2] (see also [23, Theorem 3.7]) we have

$$L_\sigma(u')(\rho) = (\sigma+i\rho)L_\sigma(u)(\rho) + 2\sigma\hat{u}(-\sigma+i\rho)$$

and

$$L_\sigma(u'')(\rho) = (\sigma+i\rho)^2L_\sigma(u)(\rho) + 4i\rho\sigma\hat{u}(-\sigma+i\rho) - 2\sigma u(0).$$

Then, (3.19) reads

$$\left[(\sigma+i\rho)^2 + \lambda(\sigma+i\rho) + (1+\hat{a}(\sigma+i\rho))A + B\hat{b}(\sigma+i\rho)\right] L_\sigma(u)(\rho) = H(\sigma,\rho) - G^{Au}_a(\sigma,\rho) - G^{Bu}_b(\sigma,\rho), \tag{3.20}$$

where $H(\sigma,\rho)$ is given by

$$H(\sigma,\rho) := -4i\rho\sigma\hat{u}(-\sigma+i\rho) + 2\sigma u(0) - 2\lambda\sigma\hat{u}(-\sigma+i\rho).$$

From (3.20) we have

$$\left[(i\rho)^2 + \lambda(i\rho) + (1+\hat{a}(i\rho))A + B\hat{b}(i\rho)\right] L_\sigma(u)(\rho) + S(\sigma,\rho)L_\sigma(u)(\rho) = H(\sigma,\rho) - G^{Au}_a(\sigma,\rho) - G^{Bu}_b(\sigma,\rho),$$

where

$$S(\sigma,\rho) = \left[(\sigma+i\rho)^2 - (i\rho)^2 + \lambda(\sigma+i\rho) - \lambda(i\rho) + (\hat{a}(\sigma+i\rho) - \hat{a}(i\rho))A + (\hat{b}(\sigma+i\rho) - \hat{b}(i\rho))B\right].$$

Since $\{i\rho\}_{\rho\in\mathbb{R}} \subset \rho_{a,b}(A,B)$ we obtain,

$$L_\sigma(u)(\rho) = H(\sigma,\rho)R(i\rho) - G^{Au}_a(\sigma,\rho)R(i\rho) - G^{Bu}_b(\sigma,\rho)R(i\rho) - S(\sigma,\rho)R(i\rho)L_\sigma(u)(\rho),$$

where $R(i\rho)$ denotes $R(i\rho) := \Big[(i\rho)^2 + \lambda(i\rho) + (1+\hat{a}(i\rho))A + B\hat{b}(i\rho)\Big]^{-1}$.

A similar argument to used in [18, Lemma A.4] shows that

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}(\hat{a}(i\rho)-\hat{a}(\sigma+i\rho))AR(i\rho)L_\sigma(u)(\rho)\phi(\rho)d\rho = 0,$$

and

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}(\hat{b}(i\rho)-\hat{b}(\sigma+i\rho))BR(i\rho)L_\sigma(u)(\rho)\phi(\rho)d\rho = 0,$$

for all $\phi\in\mathcal{S}(\mathbb{R})$. Moreover, it is easy to show that

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}\left((\sigma+i\rho)^k-(i\rho)^k\right)R(i\rho)L_\sigma(u)(\rho)\phi(\rho)d\rho = 0, \qquad \text{for } k=1,2$$

for all $\phi\in\mathcal{S}(\mathbb{R})$. Hence,

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}S(\sigma,\rho)R(i\rho)L_\sigma(u)(\rho)\phi(\rho)d\rho = 0,$$

for all $\phi\in\mathcal{S}(\mathbb{R})$. Applying the dominated convergence theorem, we have

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}H(\sigma,\rho)R(i\rho)\phi(\rho)d\rho = 0,$$

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}G_a^{Au}(\sigma,\rho)R(i\rho)\phi(\rho)d\rho = 0,$$

and

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}G_b^{Bu}(\sigma,\rho)R(i\rho)\phi(\rho)d\rho = 0,$$

for all $\phi\in\mathcal{S}(\mathbb{R})$.

Therefore, by [18, Proposition A.2.(i)] we conclude that

$$\lim_{\sigma\to 0^+}\int_{\mathbb{R}}L_\sigma(u)(\rho)\phi(\rho)d\rho = \int_{\mathbb{R}}u(\rho)\mathcal{F}(\phi)(\rho)d\rho = 0,$$

for all $\phi\in\mathcal{S}(\mathbb{R})$.

The Lemma 2.12 implies that u is constant, that is, $u(t)=x$ for all $t\in\mathbb{R}$ and some $x\in X$. We claim that $x=0$. In fact, since u is a solution to equation (3.18) we obtain $u(t)\in D(A)\cap D(B)$ and

$$0 = u''(t)+\lambda u'(t)+Au(t)+(a{*}Au)(t)+(b{*}Bu)(t) = Ax + a_0Ax + b_0Bx = (1+a_0)A + b_0B.$$

Since $0\in\rho_{a,b}(A,B)$ we obtain that $(1+a_0)A+b_0B$ is an invertible operator and therefore $x=0$, which implies that $u\equiv 0$. □

Remark 3.5. *If the Banach space X is B-convex (for example if X is a Hilbert or a UMD space), then the same consequence of Theorem 3.4 holds if we consider to the kernels a and b as* 1*-regular instead* 2*-regular kernels, because in the case, by Remark 2.7 the we just need to verify the condition* (2.7) *in order to prove that the functions N, P, Q, R and S defined in the proof of Theorem 3.4 are C^α-multipliers.*

Corollary 3.6. *In the context of Theorem 3.4, if condition (ii) is fulfilled, we have that the function u verifies u'', u' $Au, a{*}Au, b{*}Bu \in C^\alpha(\mathbb{R};X)$. Moreover, there exists a constant $C>0$ independent of $f\in C^\alpha(\mathbb{R};X)$ such that*

$$\|u''\|_{C^\alpha} + |\lambda|\,\|u'\|_{C^\alpha} + \|Au\|_{C^\alpha} + \|a{*}Au\|_{C^\alpha} + \|b{*}Bu\|_{C^\alpha} \le C\|f\|_{C^\alpha}. \tag{3.21}$$

Remark 3.7. *The inequality (3.21) is a consequence of the closed graph theorem and known as the* maximal regularity property *for equation (1.3).*

We deduce that the operator $\mathcal{F}$ defined by:

$$(\mathcal{F}u)(t) := u''(t) + \lambda u'(t) + Au(t) + (a\dot{*}Au)(t) + (b\dot{*}Bu)(t), \quad t \in \mathbb{R},$$

with domain $D(\mathcal{F}) = \mathcal{S}$ is an isomorphism onto. In fact, by Remark 3.2 we have that the space $S := C^{\alpha+2}(\mathbb{R}; X) \cap C^{\alpha}(\mathbb{R}; [D(A)]) \cap C_A^{\alpha,a}(\mathbb{R}; X) \cap C_B^{\alpha,b}(\mathbb{R}; X)$ becomes a Banach space under the norm

$$\|u\|_H := \|u''\|_{C^\alpha} + |\lambda|\, \|u'\|_{C^\alpha} + \|Au\|_{C^\alpha} + \|(a\dot{*}Au)\|_{C^\alpha} + \|(b\dot{*}Bu)\|_{C^\alpha}.$$

Such isomorphisms are crucial in the study of nonlinear evolution equations (see [1]). Indeed, define the Nemytskii's operator $N : \mathcal{S} \to C^{\alpha}(\mathbb{R}; X)$ given by $N(v)(t) = f(t, v(t))$ and the linear operator

$$\mathcal{T} := \mathcal{F}^{-1} : C^{\alpha}(\mathbb{R}; X) \to \mathcal{S}$$

by $\mathcal{T}(g) = u$ where u is the unique solution to linear problem

$$u''(t) + \lambda u'(t) + Au(t) + (a\dot{*}Au)(t) + (b\dot{*}Bu)(t) = g(t).$$

If we assume the assumption (ii) in Theorem 3.4, then the operator $\mathcal{T}$ is bounded by Corollary 3.6. To solve the semilinear problem

$$u''(t) + \lambda u'(t) + Au(t) + (a\dot{*}Au)(t) + (b\dot{*}Bu)(t) = f(t, u(t)), \quad t \in \mathbb{R} \tag{3.22}$$

we need to show that the operator $\mathcal{R} : \mathcal{S} \to \mathcal{S}$ defined by $\mathcal{R} = \mathcal{T}N$ has a fixed point. For more details, we refer to Amann [1, 2].

4. Existence of Mild solutions on the real line

Observe that by Corollary 3.6, the solution u to equation (1.3) is twice differentiable and has certain regularity. However, in more general conditions it is interesting to study the existence of solutions to (1.3) without this regularity. In this section, we introduce a concept of *mild solution* to equation (1.3) and we give necessary conditions for the existence and uniqueness.

We define the functions g_1 and g_2 respectively by $g_1(t) = 1$ and $g_2(t) = t$ for all $t \in \mathbb{R}$. The usual convolution between the functions f and g, denoted by $(f * g)(t)$, is defined by

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds,$$

for all $t \in \mathbb{R}$. Observe that

$$(g_1 * f)(t) = \int_0^t f(s)ds \quad \text{and} \quad (g_2 * f)(t) = \int_0^t (t-s)f(s)ds,$$

and $(g_2 * f)(t) = (g_1 * g_1 * f)(t)$ for all $t \in \mathbb{R}$. By $\mathrm{BUC}(\mathbb{R}, X)$ we denote the space of all bounded and uniformly continuous functions on $\mathbb{R}$ with values in X equipped with the norm $\|\cdot\|_\infty$.

Definition 4.8. *Let $f \in \mathrm{BUC}(\mathbb{R}, X)$. A function $u \in \mathrm{BUC}(\mathbb{R}, X)$ is called a* mild solution *to (1.3) if $(g_2 * u)(t), (g_2 * (a\dot{*}u))(t) \in D(A)$, $(g_2 * (b\dot{*}u))(t) \in D(B)$, for all $t \in \mathbb{R}$ and there exists $y \in X$ such that*

$$\begin{aligned} u(t) &= u(0) + ty + \lambda t u(0) - \lambda(g_1 * u)(t) - A(g_2 * u)(t) - A(g_2 * (a\dot{*}u))(t) \\ &- B(g_2 * (b\dot{*}u))(t) + (g_2 * f)(t), \end{aligned} \tag{4.23}$$

for all $t \in \mathbb{R}$.

We notice that the vector y in this definition is unique. Observe that if $a(t) = b(t) = 0$, for all $t \in \mathbb{R}$ and $\lambda = 0$, then this concept of mild solution is the same that in case of the second order problem $u''(t) + Au(t) = f(t)$, $t \in \mathbb{R}$, see [27].

Now, we consider the problem of the existence and uniqueness of mild solution to equation (1.3) on the real line. On the space $\mathrm{BUC}(\mathbb{R}, X)$ we define the linear operator $\mathcal{L} : \mathrm{BUC}(\mathbb{R}, X) \to \mathrm{BUC}(\mathbb{R}, X)$ which takes a function $f \in \mathrm{BUC}(\mathbb{R}, X)$ into the solution $u \in \mathrm{BUC}(\mathbb{R}, X)$ of equation (1.3). If such solution u is unique for each function f, then by the closed graph theorem $\mathcal{L}$ is a bounded operator. Moreover, we notice that if the mild solution u is twice differentiable, that is, $u \in C^2(\mathbb{R}, X)$, then u is a classical solution to (1.3).

The next result gives necessary conditions for the existence and uniqueness of mild solutions to (1.3). Its proof follows similarly to [27, Theorem 2.5].

Theorem 4.9. *Let $a, b \in L^1(\mathbb{R}_+)$. Let $A : D(A) \subset X \to X$ and $B : D(B) \subset X \to X$ be closed linear operators defined in a Banach space X with $D(A) \cap D(B) \neq \{0\}$. Assume that for every $f \in \mathrm{BUC}(\mathbb{R}, X)$ there exists a unique mild solution $u \in \mathrm{BUC}(\mathbb{R}, X)$ to equation (1.3). Then $i\eta \in \rho_{a,b}(A, B)$ for all $\eta \in \mathbb{R}$, and there exists a constant M such that $\| \left[(i\eta)^2 + \lambda(i\eta) + (1 + a_\eta)A + b_\eta B\right]^{-1} \| \leq M$ for all $\eta \in \mathbb{R}$.*

Proof. We first prove that $[(i\eta)^2 + \lambda(i\eta) + (1 + a_\eta)A + b_\eta B]$ is surjective. We take arbitraries $\eta \in \mathbb{R}$ and $y \in X$. For $s, t \in \mathbb{R}$, we define the function $f_s(t) := e^{i\eta(t+s)}y = e^{i\eta s}f_0(t) = f_0(t + s)$ where $f_0(t) := e^{i\eta t}y$. Since $f_s \in \mathrm{BUC}(\mathbb{R}, X)$ there exists a unique mild solution $u_s \in \mathrm{BUC}(\mathbb{R}, X)$ to (1.3). We claim that

$$u_s(t) = e^{i\eta s}u_0(t) = u_0(s + t) \tag{4.24}$$

for all $s, t \in \mathbb{R}$. In fact, since u_s is a mild solution to equation (1.3) with f_s, there exists $y_s \in X$ such that

$$\begin{aligned} u_s(t) &= u_s(0) + ty_s + \lambda t u_s(0) - \lambda(g_1 * u_s)(t) - A(g_2 * u_s)(t) - A(g_2 * (a \dot{*} u_s))(t) \\ &\quad -B(g_2 * (b \dot{*} u_s))(t) + (g_2 * f_s)(t), \end{aligned} \tag{4.25}$$

for all $t \in \mathbb{R}$. Multiplying both sides by $e^{-i\eta s}$ we obtain

$$\begin{aligned} e^{-i\eta s}u_s(t) &= e^{-i\eta s}u_s(0) + te^{-i\eta s}y_s + \lambda t e^{-i\eta s}u_s(0) - \lambda \int_0^t e^{-i\eta s}u_s(r)dr - A\int_0^t (t-r)e^{-i\eta s}u_s(r)dr \\ &\quad -A\int_0^t (t-r)e^{-i\eta s}(a \dot{*} u_s)(r)dr - B\int_0^t (t-r)e^{-i\eta s}(b \dot{*} u_s)(r)dr + \int_0^t (t-r)e^{-i\eta s}f_s(r)dr. \end{aligned}$$

Then, $e^{-i\eta s}u_s(t)$ is a mild solution to (1.3) with f_0, because

$$e^{-i\eta s}(a \dot{*} u_s)(r) = \int_{-\infty}^r a(r-w)e^{-i\eta s}u_s(w)dw \quad \text{and} \quad \int_0^t (t-r)e^{-i\eta s}f_s(r)dr = \int_0^t (t-r)f_0(r)dr.$$

From the uniqueness, we obtain $e^{-i\eta s}u_s(t) = u_0(t)$ for all $s, t \in \mathbb{R}$ and thus we get the first equality in (4.24).

On the other hand, since u_0 is a mild solution of (1.3) with f_0, where exists $y_0 \in X$ such that

$$\begin{aligned} u_0(t) &= u_0(0) + ty_0 + \lambda t u_0(0) - \lambda(g_1 * u_0)(t) - A(g_2 * u_0)(t) - A(g_2 * (a \dot{*} u_0))(t) \\ &\quad - B(g_2 * (b \dot{*} u_0))(t) + (g_2 * f_0)(t), \end{aligned}$$

for all $t \in \mathbb{R}$. Then

$$\begin{aligned} u_0(s+t) &= u_0(0) + (s+t)y_0 + \lambda(s+t)u_0(0) - \lambda(g_1 * u_0)(s+t) \\ &\quad -A(g_2 * u_0)(s+t) - A(g_2 * (a \dot{*} u_0))(s+t) - B(g_2 * (b \dot{*} u_0))(s+t) + (g_2 * f_0)(s+t), \end{aligned}$$

and

$$\begin{aligned} u_0(s) &= u_0(0) + sy_0 + \lambda s u_0(0) - \lambda(g_1 * u_0)(s) - A(g_2 * u_0)(s) - A(g_2 * (a \dot{*} u_0))(s) \\ &\quad - B(g_2 * (b \dot{*} u_0))(s) + (g_2 * f_0)(s), \end{aligned}$$

which implies

$$\begin{aligned} u_0(s+t) - u_0(s) &= ty_0 + \lambda t u_0(0) - \lambda[(g_1 * u_0)(s+t) - (g_1 * u_0)(s)] - A[(g_2 * u_0)(s+t) - (g_2 * u_0)(s)] \\ &\quad - A[(g_2 * (a \dot{*} u_0))(s+t) - (g_2 * (a \dot{*} u_0))(s)] - B[(g_2 * (b \dot{*} u_0))(s+t) - (g_2 * (b \dot{*} u_0))(s)] \\ &\quad + [(g_2 * f_0)(s+t) - (g_2 * f_0)(s)]. \end{aligned} \tag{4.26}$$

From (4.25) and (4.26) we have

$$\begin{aligned}[u_s(t)-u_0(s+t)] &= [u_s(0)-u_0(s)]+t[y_s-y_0]+\lambda t[u_s(0)-u_0(0)] \\ &\quad -\lambda[(g_1*u_s)(t)-(g_1*u_0)(s+t)+(g_1*u_0)(s)] \\ &\quad -A[(g_2*u_s)(t)-(g_2*u_0)(s+t)+(g_2*u_0)(s)] \\ &\quad -A[(g_2*(a\dot{*}u_s))(t)-(g_2*(a\dot{*}u_0))(s+t)+(g_2*(a\dot{*}u_0))(s)] \\ &\quad -B[(g_2*(b\dot{*}u_s))(t)-(g_2*(b\dot{*}u_0))(s+t)+(g_2*(b\dot{*}u_0))(s)] \\ &\quad +[(g_2*f_s)(t)-(g_2*f_0)(s+t)+(g_2*f_0)(s)]. \end{aligned} \tag{4.27}$$

Let $U(t):=u_s(t)-u_0(s+t)$. Easy computations show that

$$(g_1*u_s)(t)-(g_1*u_0)(s+t)+(g_1*u_0)(s)=(g_1*U)(t),$$

$$A[(g_2*u_s)(t)-(g_2*u_0)(s+t)+(g_2*u_0)(s)]=A(g_2*U)(t)-tA(g_1*u_0)(s),$$

$$A[(g_2*(a\dot{*}u_s))(t)-(g_2*(a\dot{*}u_0))(s+t)+(g_2*(a\dot{*}u_0))(s)]=A(g_2*(a\dot{*}U))(t)-tA(g_1*(a\dot{*}u_0))(s)$$

(and analogously for the operator B and the kernel b) and

$$[(g_2*f_s)(t)-(g_2*f_0)(s+t)+(g_2*f_0)(s)]=-t(g_1*f_0)(s).$$

From (4.27) we obtain

$$\begin{aligned} U(t) &= U(0)+t\big[y_s-y_0-\lambda(u_0(s)+u_0(0))+A(g_1*u_0)(s)+A(g_1*(a\dot{*}u_0))(s)+B(g_1*(b\dot{*}u_0))(s) \\ &\quad -(g_1*f_0)(s)\big]+\lambda tU(0)-\lambda(g_1*U)(t)-A(g_2*U)(t)-A(g_2*(a\dot{*}U))(t)-B(g_2*(b\dot{*}U))(t). \end{aligned}$$

Therefore, U is a mild solution to the homogeneous equation $u''(t)+\lambda u'(t)+Au(t)+(a\dot{*}Au)(t)+(b\dot{*}Bu)(t)=0$. By uniqueness, we conclude that $U(t)=0$ for all $t\in\mathbb{R}$ and therefore $u_s(t)=u_0(s+t)$. The claim is proved.

Now, we take $x=u_0(0)$. By the claim, we have $u_0(t)=u_0(0+t)=u_0(t+0)=e^{i\eta t}u_0(0)=e^{i\eta t}x$, that is, $u_0(t)=e^{i\eta t}x$. Note that $u_0(\cdot)\in C^2(\mathbb{R},X)$ and therefore u is a classical solution of (1.3) with $f_0(t)$, that is

$$u_0''(t)+\lambda u_0'(t)+Au_0(t)+(a\dot{*}Au_0)(t)+(b\dot{*}Bu_0)(t)=f_0(t)$$

for all $t\in\mathbb{R}$. In particular, if $t=0$ then $x\in D(A)\cap D(B)$ and we obtain

$$e^{i\eta t}[(i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B]x=f_0(0)=y,$$

which implies that $[(i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B]$ is surjective for all $\eta\in\mathbb{R}$.

In order to prove the injectivity, let $\eta\in\mathbb{R}$ and suppose that for $x\in D(A)\cap D(B)$

$$[(i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B]x=0. \tag{4.28}$$

Let $u(t)=e^{i\eta t}x$. Then, u is a classical solution (and then a mild solution) to (1.3) with $f\equiv 0$, because $(a\dot{*}Au)(t)=e^{i\eta t}a_\eta Ax$ and $(b\dot{*}Bu)(t)=e^{i\eta t}b_\eta Bx$. From (4.28) we obtain

$$u''(t)+\lambda u'(t)+Au(t)+(a\dot{*}Au)(t)+(b\dot{*}Bu)(t)=e^{i\eta t}[(i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B]x=0. \tag{4.29}$$

and from the uniqueness it follows that $u(t)=0$ for all $t\in\mathbb{R}$ and thus $x=0$. Therefore, $[(i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B]$ is injective.

Finally, we take arbitrary $\eta\in\mathbb{R}$ and $x\in X$. Define $y:=[(i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B]^{-1}x$. Then $u_0(t)=e^{i\eta t}y$ is a classical solution to (1.3) with $f_0(t)=e^{i\eta t}x$, because

$$u''(t)+\lambda u'(t)+Au(t)+(a\dot{*}Au)(t)+(b\dot{*}Bu)(t)=e^{i\eta t}[(i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B]y=e^{i\eta t}x=f_0(t).$$

On the other hand, observe that $\|[(i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B]^{-1}x\|_X=\|y\|_X=\|u_0\|_\infty$ and $\|x\|_X=\|f_0\|_\infty$. Since the linear operator $\mathcal{L}$ is bounded we obtain

$$\|[(i\eta)^2+\lambda(i\eta)+(1+a_\eta)A+b_\eta B]^{-1}x\|_X=\|y\|_X=\|u_0\|_\infty=\|\mathcal{L}f_0\|_\infty\le\|\mathcal{L}\|\,\|f_0\|_\infty=\|\mathcal{L}\|\,\|x\|_X.$$

Therefore, there exists a constant M such that

$$\| \left[(i\eta)^2 + \lambda(i\eta) + (1+a_\eta)A + b_\eta B\right]^{-1} \| \leq M,$$

for all $\eta \in \mathbb{R}$. □

5. Examples

In this section we consider some applications of the results presented in the previous sections. We first consider the second order equation

$$u''(t) + \lambda u'(t) + Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + \int_{-\infty}^{t} b(t-s)u(s)ds = f(t), \quad t \in \mathbb{R}, \tag{5.30}$$

where $\lambda \in \mathbb{R}$, A is self-adjoint dissipative operator defined in a Hilbert space H, the kernels $a, b \in L^1(\mathbb{R}_+)$ are 2-regular and $f \in C^\alpha(\mathbb{R}, H)$. We recall that a_η and b_η denote $a_\eta = \hat{a}(\eta)$ and $b_k = \hat{b}(\eta)$ and we assume that $a_\eta \neq 1$ for all $\eta \in \mathbb{R}$.

Observe that if $B = I$, that is, B is the identity operator in H, then

$$\big((i\eta)^2 + \lambda(i\eta) + (1+a_\eta)A + b_\eta I\big)^{-1} = \frac{-1}{1+a_\eta}\left(\frac{(i\eta)^2 - \lambda(i\eta) - b_\eta}{1+a_\eta} - A\right)^{-1} \tag{5.31}$$

for all $\eta \in \mathbb{R}$. For each $\eta \in \mathbb{R}$, we write $\mu_\eta := \frac{(i\eta)^2 - \lambda(i\eta) - b_\eta}{1+a_\eta}$ and we suppose that $\mu_\eta \notin \sigma(A)$ for all $\eta \in \mathbb{R}$. Suppose that $\mathrm{Im}(\mu_\eta) \neq 0$ for all $\eta \in \mathbb{R}$. Since A is a self-adjoint dissipative operator, then A is sectorial operator with $\sigma(A) \subset (-\infty, 0]$ and therefore there exists a constant M such that

$$\|\mu_\eta(\mu_\eta - A)^{-1}\| \leq M \tag{5.32}$$

for all $\eta \in \mathbb{R}$.

Proposition 5.10. *Assume the above conditions. Suppose that* $\mathrm{Im}(\mu_\eta) \neq 0$ *for all* $\eta \in \mathbb{R}$. *If* $f \in C^\alpha(\mathbb{R}, H)$, *then the equation* (5.30) *is* C^α*-well posed.*

Proof. According to Theorem 3.4 we need to prove that $\sup_{\eta\in\mathbb{R}} \|(i\eta)^2 N_\eta\| < \infty$, $\sup_{\eta\in\mathbb{R}} \|a_\eta A N_\eta\| < \infty$ and $\sup_{\eta\in\mathbb{R}} \|b_\eta N_\eta\| < \infty$, where $N_\eta := \big((i\eta)^2 + \lambda(i\eta) + (1+a_\eta)A + b_\eta I\big)^{-1}$.

In fact, since $(1+a_\eta)N_\eta = -(\mu_\eta - A)^{-1}$ and $A(\mu_\eta - A)^{-1} = \mu_\eta(\mu_\eta - A)^{-1} - I$ we obtain by (5.32) that

$$\|(i\eta)^2 N_\eta\| = \frac{|(i\eta)^2|}{|1+a_\eta|}\|(1+a_\eta)N_\eta\| = \frac{|\eta|^2}{|1+a_\eta|}\|(\mu_\eta - A)^{-1}\| \leq \frac{|\eta|^2}{|1+a_\eta|}\frac{M}{|\mu_\eta|} = \frac{M|\eta|^2}{|(i\eta)^2 - \lambda(i\eta) - b_\eta|},$$

which is uniformly bounded. Therefore, $\sup_{\eta\in\mathbb{R}} \|(i\eta)^2 N_\eta\| < \infty$.

On the other hand, the Riemann-Lebesgue lemma implies that $\frac{|a_\eta|}{|1+a_\eta|}$ is bounded and therefore

$$\|a_\eta A N_\eta\| = \frac{|a_\eta|}{|1+a_\eta|}\|(1+a_\eta)AN_\eta\| = \frac{|a_\eta|}{|1+a_\eta|}\|A(\mu_\eta - A)^{-1}\| \leq C\left(1 + \|\mu_\eta(\mu_\eta - A)^{-1}\|\right) \leq C(1+M),$$

for all $\eta \in \mathbb{R}$. We conclude that $\sup_{\eta\in\mathbb{R}} \|a_\eta A N_\eta\| < \infty$. Finally, we have

$$\|b_\eta N_\eta\| = \frac{|b_\eta|}{|1+a_\eta|}\|(1+a_\eta)N_\eta\| = \frac{|b_\eta|}{|1+a_\eta|}\|(\mu_\eta - A)^{-1}\| \leq \frac{|b_\eta|}{|1+a_\eta|}\frac{M}{|\mu_\eta|} \leq \frac{M|b_\eta|}{|(i\eta)^2 - \lambda(i\eta) - b_\eta|},$$

which is uniformly bounded by the Riemann-Lebesgue lemma. Thus $\sup_{\eta\in\mathbb{R}} \|b_\eta N_\eta\| < \infty$.

We conclude by Theorem 3.4 that (5.30) is C^α-well posed, which means that for every $f \in C^\alpha(\mathbb{R}, H)$, there exists a unique solution $u \in \mathcal{S}$ of equation (5.30). Moreover, by Corollary 3.6 the function u verifies $u'', u', Au, (a{*}Au), (b{*}u) \in C^\alpha(\mathbb{R}, H)$. □

Now, for $\alpha, \beta > 0$ we consider the following problem

$$\left\{ \begin{array}{l} u''(t,x) = -\alpha \displaystyle\int_{-\infty}^{t} e^{-\beta(t-s)} \Delta u(s,x) ds + f(t,x), \quad t \in \mathbb{R}, \\ \qquad u = 0 \quad \text{in } \mathbb{R} \times \partial\Omega, \end{array} \right. \tag{5.33}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Clearly, the kernel $b(t) = \alpha e^{-\beta t}$ is 2-regular (see [26, Proposition 3.3]) and $b_\eta = \hat{b}(\eta) = \frac{\alpha}{\beta + i\eta}$ for all $\eta \in \mathbb{R}$. Let $\alpha_\eta := \mathrm{Re}(b_\eta) = \frac{\alpha\beta}{\beta^2+\eta^2}$ and $\beta_\eta := \mathrm{Im}(b_\eta) = \frac{-\alpha\eta}{\beta^2+\eta^2}$.

Now, we notice that

$$\|(i\eta)^2((i\eta)^2 I + b_\eta \Delta)^{-1}\| = \frac{|\eta|^2}{|b_\eta|} \left\| \left(\frac{\eta^2}{b_\eta} I - \Delta \right)^{-1} \right\|.$$

If we take $X = H^{-1}(\Omega)$, then by [14, p. 74], there exists a constant $M > 0$ such that

$$\|(zI - \Delta)^{-1}\| \leq \frac{M}{1 + |z|}$$

whenever $\mathrm{Re} z \geq -c(1 + |\mathrm{Im} z|)$, where $c > 0$ is certain constant. If $z = \eta^2/b_\eta$ then the inequality $\mathrm{Re} z \geq -c(1 + |\mathrm{Im} z|)$ is equivalent to

$$\alpha_\eta \geq -c(\alpha_\eta^2 + \beta_\eta^2 + |\beta_\eta|) \tag{5.34}$$

for all $\eta \in \mathbb{R}$. Since $\alpha, \beta > 0$, then the (5.34) holds with $c = 1$. Hence

$$\left\| \left(\frac{\eta^2}{b_\eta} I - \Delta \right)^{-1} \right\| \leq \frac{M}{1 + \left| \frac{\eta^2}{b_\eta} \right|},$$

which implies

$$\frac{|\eta^2|}{|b_\eta|} \left\| \left(\frac{\eta^2}{b_\eta} I - \Delta \right)^{-1} \right\| \leq M \quad \text{and} \quad \left\| \left(\frac{\eta^2}{b_\eta} I - \Delta \right)^{-1} \right\| \leq M,$$

for all $\eta \in \mathbb{R}$. We conclude that

$$\sup_{\eta \in \mathbb{R}} \|(i\eta)^2((i\eta)^2 I + b_\eta \Delta)^{-1}\| < \infty.$$

On the other hand, since

$$\|b_\eta((i\eta)^2 I + b_\eta \Delta)^{-1}\| = \left\| \left(\frac{\eta^2}{b_\eta} I - \Delta \right)^{-1} \right\|,$$

we obtain

$$\sup_{\eta \in \mathbb{R}} \|b_\eta((i\eta)^2 I + b_\eta \Delta)^{-1}\| < \infty.$$

By Theorem 3.4 we conclude that (5.33) is C^α-well posed.

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