HÖLDER CONTINUOUS SOLUTIONS FOR SOBOLEV TYPE DIFFERENTIAL EQUATIONS

RODRIGO PONCE

ABSTRACT. We characterize existence and uniqueness of solutions of an abstract differential equation in Hölder spaces.

1. INTRODUCTION

Let A and M be two closed linear operators defined on a Banach space X with domains $D(A)$ and $D(M)$, respectively. In this paper, we study the existence, uniqueness and maximal regularity of solutions for the following Sobolev (or degenerate) type differential equation

$$\frac{d}{dt}(Mu(t)) = Au(t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $f \in C^{\alpha}(\mathbb{R}; X)$, $0 < \alpha < 1$, and the domains of A and M satisfy $D(A) \cap D(M) \neq \{0\}$.

A large number of partial differential equations arising in physics and applied sciences can be expressed by the model (1.1). For example, if $A = \Delta$ is the Laplacian and $M = m$ is the multiplication operator by a function $m(x)$, then the model (1.1) describes the infiltration of water in unsaturated porous media, in which saturation might occur. See [9] and [22] for more details.

A detailed study of linear abstract Sobolev (or degenerate) type differential equations (1.1), has been described in the monographs by Favini and Yagi [14] and by Sviridyuk and Fedorov [25]. See moreover [19].

Existence and uniqueness of Hölder continuous solutions for equations in the form of (1.1) have been extensively studied in the literature. See [11, 12, 13, 14, 15, 16, 25] and the references therein. The obtained results give sufficient conditions for the existence and uniqueness of Hölder solutions to equation (1.1), but leave as an open problem to *characterize* the well-posedness (or maximal regularity) in terms of hypothesis in the *modified resolvent operator* $(\lambda M - A)^{-1}$ of the operators M and A. We notice that, the problem of characterize the well-posedness (or maximal regularity) of abstract equations on Hölder spaces have been studied intensively in the last years. See e.g. [8, 10, 18, 20, 23].

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We notice that with the change of variable $v(t) = Mu(t)$ we reduce the problem (1.1) to the multivalued differential equation

$$v'(t) \in Lv(t) + f(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where $L = AM^{-1}$ and $D(L) = M(D(A))$. Then, formally the equation (1.1) reduces to the equation of first order studied in [2]. However, is necessary verify that all the steps in [2] to the single-valued case are valid for the multivalued case (1.2).

In some previous works, to establish the existence and uniqueness of solutions to equation (1.1) some assumptions on operators A and M are considered:

i) $D(A) \subset D(M)$ and A admits a continuous inverse operator A^{-1} [11, 12],
ii) $D(A) \subset D(M)$ and M has the bounded inverse [14],
iii) $D(M) \subset D(A)$ and M has the compact inverse [4, 5].

However, these conditions on operators A and M are restrictive. On the other hand, Arendt, Batty and Bu [2], using operator-valued Fourier multiplier theorems, have derived spectral characterizations of well-posedness of the equation (1.1) on Hölder spaces in the case when $M = I$, is the identity operator. This connection motivates the question of whether it is possible to obtain a similar characterization to that given in [21] in the periodic case, for the equation (1.1) in the case of the class of Hölder spaces $C^\alpha(\mathbb{R}; X)$, $0 < \alpha < 1$.

In this paper, we study existence and uniqueness solutions to equation (1.1) in the Hölder spaces $C^\alpha(\mathbb{R}; X)$ $(0 < \alpha < 1)$, without assuming M has bounded (or compact) inverse as well without any assumption on the relation between $D(A)$ and $D(M)$.

The paper is organized as follows. Section 2 collects the preliminaries and some results about the operator-valued Fourier multipliers in Hölder spaces. Section 3 is devoted to the main result, where the well-posedness of equation (1.1) and some consequences are studied. We remark that in these results, there is not conditions in commutativity of A with M, or in the existence of bounded inverse of A or M. In Section 4, some examples are examined.

2. Preliminaries

Let X and Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ be the space of all bounded linear operators from X to Y. If $X = Y$, we write simply $\mathcal{B}(X)$. Let $0 < \alpha < 1$. We denote by $C^\alpha(\mathbb{R}; X)$ the space of all X-valued functions f on $\mathbb{R}$, such that

$$||f||_\alpha = \sup_{t \neq s} \frac{||f(t) - f(s)||}{|t - s|^\alpha} < \infty.$$

If we define $||f||_{C^\alpha} := ||f||_\alpha + ||f(0)||$, then $C^\alpha(\mathbb{R}; X)$ is a Banach space under the norm $|| \cdot ||_{C^\alpha}$.

The kernel of the seminorm $|| \cdot ||_\alpha$ on $C^\alpha(\mathbb{R}; X)$ is the space of all constant functions and the corresponding quotient space $\dot{C}^\alpha(\mathbb{R}; X)$ is a Banach space in the induced norm. We identify a function $f \in C^\alpha(\mathbb{R}; X)$ with its equivalence class

$$\dot{f} := \{g \in C^\alpha(\mathbb{R}; X) : f - g \equiv \text{ constant}\}.$$

In this way, $\dot{C}^\alpha(\mathbb{R};X)$ may be identified with the space of all $f \in C^\alpha(\mathbb{R};X)$ such that $f(0)=0$. See [2, Section 5].

We also consider in this paper, the Banach space $C^{\alpha+1}(\mathbb{R};X)$, which consists of all $u \in C^1(\mathbb{R};X)$ such that $u' \in C^\alpha(\mathbb{R};X)$ with the norm

$$||u||_{C^{\alpha+1}} = ||u'||_{C^\alpha} + ||u(0)||.$$

We denote by $\mathcal{F}f$, the Fourier transform of f, that is

$$(\mathcal{F}f)(s) := \int_{\mathbb{R}} e^{-ist} f(t)dt,$$

for $s \in \mathbb{R}$ and $f \in L^1(\mathbb{R};X)$.

We use the symbol $\hat{f}(\lambda)$ for the *Carleman transform of* f :

$$\hat{f}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} f(t), \ \mathrm{Re}\lambda > 0, \\ \\ -\int_{-\infty}^0 e^{-\lambda t} f(t), \ \mathrm{Re}\lambda < 0, \end{cases}$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R};X)$ is of *subexponential growth*, by this we mean

$$\int_{-\infty}^{\infty} e^{-\epsilon|t|} ||f(t)||dt < \infty, \qquad \text{for each } \epsilon > 0.$$

Let $\Omega \subset \mathbb{R}$ be an open set. By $C_c^\infty(\Omega)$ we denote the space of all $C^\infty-$functions in Ω having compact support in Ω.

Definition 2.1. *Let $N : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X,Y)$ be continuous. We say that N is a $\dot{C}^\alpha-$multiplier if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R};X) \to \dot{C}^\alpha(\mathbb{R};Y)$ such that*

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot N))(s) f(s)ds \tag{2.1}$$

for all $f \in C^\alpha(\mathbb{R};X)$ and all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Here $(\mathcal{F}(\phi \cdot N))(s) = \int_{\mathbb{R}} e^{-ist}\phi(t)N(t)dt \in \mathcal{B}(X,Y)$. Observe that the right-hand side of (2.1) does not depend on the representative of $\dot{f}$ since

$$\int_{\mathbb{R}} (\mathcal{F}(\phi N)(s))(s)ds = 2\pi(\phi N)(0) = 0.$$

Therefore, if L exists, then it is well defined. Moreover, left-hand side of (2.1) determines the function $Lf \in C^\alpha(\mathbb{R};X)$ uniquely up to some constant (by [2, Lemma 5.1]). Moreover, if (2.1) holds, then $L : \dot{C}^\alpha(\mathbb{R};X) \to \dot{C}^\alpha(\mathbb{R};Y)$ is linear and continuous (see [2, Definition 5.2]) and if $f \in C^\alpha(\mathbb{R};X)$ is bounded, then Lf is bounded as well (see [2, Remark 6.3]).

The following multiplier theorem is due to Arendt, Batty and Bu [2, Theorem 5.3].

Theorem 2.2. [2] *Let $N \in C^2(\mathbb{R} \setminus \{0\}; \mathcal{B}(X,Y))$ be such that*

$$\sup_{t\neq 0} ||N(t)|| + \sup_{t \neq 0} ||tN'(t)|| + \sup_{t \neq 0} ||t^2 N''(t)|| < \infty. \tag{2.2}$$

Then, N is a $\dot{C}^\alpha-$multiplier.

Example 2.3. *Let X be an Banach space and $0<\alpha<1$. Define $N(t)=I$ for $t\geq 0$ and $N(t)=0$ for $t<0$. It follows from Theorem 2.2 that N is a $\dot{C}^{\alpha}$-multiplier. The associated operator on $\dot{C}^{\alpha}(\mathbb{R};X)$ is called the* Riesz projection.

Example 2.4. *Let X be an Banach space and $0<\alpha<1$. Define $N(t)=(-i\,\mathrm{sign}t)I$ for $t\in\mathbb{R}$. Then N is a $\dot{C}^{\alpha}$-multiplier by Theorem 2.2. The associated operator on $\dot{C}^{\alpha}(\mathbb{R};X)$ is called the* Hilbert transform.

Recall that a Banach space X has the *Fourier type* p, with $1\leq p\leq 2$, if the Fourier transform defines a bounded linear operator from $L^p(\mathbb{R};X)$ to $L^q(\mathbb{R};X)$, where $1/p+1/q=1$. As examples, the $L^p(\Omega)$, with $1\leq p\leq 2$ has Fourier type p; the Banach space X has the Fourier type 2 if and only if X is isomorphic to a Hilbert space; X has Fourier type p if and only if X^* has Fourier type p. Every Banach space has Fourier type 1. A Banach space X is said to be $B-$convex if it has Fourier type p, for some $p>1$. Every uniformly convex space is $B-$convex. For more details of B-convex spaces, see for instance [17].

Remark 2.5.

If X is $B-$convex, in particular if X is a UMD space, then the Theorem 2.2 holds if the condition (2.2) is replaced by the weaker condition

$$\sup_{t\neq 0}||N(t)||+\sup_{t\neq 0}||tN'(t)||<\infty, \tag{2.3}$$

where $N\in C^1(\mathbb{R}\setminus\{0\};\mathcal{B}(X,Y))$, see [2, Remark 5.5].

We conclude this section with two Lemmas.

Lemma 2.6. [2] *Let $f\in C^{\alpha}(\mathbb{R};X)$. Then f is constant if and only if $\int_{\mathbb{R}}f(s)(\mathcal{F}\varphi)(s)ds=0$ for all $\varphi\in C_c^{\infty}(\mathbb{R}\setminus\{0\})$.*

Define $\mathrm{id}:\mathbb{R}\to\mathbb{C}$ by $\mathrm{id}(s)=is$.

Lemma 2.7. [2] *Let $0<\alpha<1$, $u,v\in C^{\alpha}(\mathbb{R};X)$. Then, the following assertions are equivalent,*

(i) $u\in C^{\alpha+1}(\mathbb{R};X)$ and $u'-v$ is constant;

(ii) $\int_{\mathbb{R}}v(s)\mathcal{F}(\phi)(s)ds=\int_{\mathbb{R}}u(s)\mathcal{F}(\mathrm{id}\cdot\phi)(s)ds$, for all $\phi\in C_c^{\infty}(\mathbb{R}\setminus\{0\})$.

3. A characterization

Let X a Banach space. In this section, we consider the degenerate differential equation

$$\frac{d}{dt}(Mu(t))=Au(t)+f(t),\qquad t\in\mathbb{R}, \tag{3.1}$$

where $A:D(A)\subseteq X\to X$ and $M:D(M)\subseteq X\to X$ are closed linear operators defined on X, with $D(A)\cap D(M)\neq\{0\}$, and $f\in C^{\alpha}(\mathbb{R};X)$, $0<\alpha<1$.

The *M-modified resolvent set of A*, $\rho_M(A)$, is defined by

$$\rho_M(A):=\{\lambda\in\mathbb{C}:(\lambda M-A):D(A)\cap D(M)\to X$$
$$\text{is invertible and }(\lambda M-A)^{-1}\in\mathcal{B}(X,[D(A)\cap D(M)])\}.$$

We define the set

$$H^{1,\alpha}(\mathbb{R};[D(M)]) = \{u \in C^\alpha(\mathbb{R};[D(M)]) : \exists v \in C^\alpha(\mathbb{R};X) \text{ such that } v = (Mu)'\}.$$

We establish the definition of *maximal regularity* or *well-posedness* of problem (3.1) as follows.

Definition 3.1. *We say that the equation (3.1) is* C^α-well posed *if, for each* $f \in C^\alpha(\mathbb{R};X)$, *there exists a unique function* $u \in C^\alpha(\mathbb{R};[D(A)\cap D(M)]) \cap H^{1,\alpha}(\mathbb{R},[D(M)])$ *such that the equation (3.1) holds for all* $t \in \mathbb{R}$.

Remark 3.2.

Observe that if (3.1) is C^α-well posed, it follows from the closed graph theorem that the map $L : C^\alpha(\mathbb{R};X) \to C^\alpha(\mathbb{R};[D(A)\cap D(M)]) \cap H^{1,\alpha}(\mathbb{R};[D(M)])$ which associates to f the unique solution u of (3.1) is linear and continuous. Indeed, since A, M are closed operators, we have that the space $H := C^\alpha(\mathbb{R};[D(A)\cap D(M)])\cap H^{1,\alpha}(\mathbb{R};[D(M)])$ endowed with the norm

$$||u||_H := ||(Mu)'||_{C^\alpha} + ||Au||_{C^\alpha} + ||u||_{C^\alpha}$$

is a Banach space.

We begin with the following result.

Proposition 3.3. *Let* $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ *closed linear operators defined on a Banach space* X *satisfying* $D(A)\cap D(M) \neq \{0\}$. *Suppose that the problem (3.1) is* C^α*-well posed. Then,*

(*i*) $i\mathbb{R} \subset \rho_M(A)$,

(*ii*) $\sup_{\eta\in\mathbb{R}} ||i\eta M(i\eta M - A)^{-1}|| < \infty$.

Proof. Let $\eta \in \mathbb{R}$ and suppose that $(i\eta M - A)x = 0$ where $x \in D(A)\cap D(M)$. Let $u(t) = e^{i\eta t}x$. Then, u is a solution to (3.1) with $f \equiv 0$. Hence, by uniqueness it follows that $u \equiv 0$, that is, $x = 0$. We conclude that $(i\eta M - A)$ is injective. In order to show the surjectivity, let $y \in X$. Let $L : C^\alpha(\mathbb{R};X) \to C^\alpha(\mathbb{R};[D(A)\cap D(M)]) \cap H^{1,\alpha}(\mathbb{R};[D(M)])$ be the bounded linear operator which takes each $f \in C^\alpha(\mathbb{R};X)$ to the unique solution u of equation (3.1). Let $\eta \in \mathbb{R}$, $f(t) = e^{i\eta t}y$ and $u = Lf$. Then, for fixed $s \in \mathbb{R}$ we have that $v_1(t) := u(t+s)$ and $v_2(t) := e^{i\eta s}u(t)$ are both solutions of (3.1) with $g(t) = e^{i\eta s}f(t)$. Hence, $v_1 = v_2$, that is, $u(t+s) = e^{is\eta}u(t)$ for all $s,t \in \mathbb{R}$. Let $x = u(0) \in D(A)\cap D(M)$. Then, $u(t) = e^{i\eta t}x$ satisfies the equation (3.1) for all $t \in \mathbb{R}$, in particular, for $t = 0$, we obtain,

$$(i\eta M - A)x = i\eta Mu(0) - Au(0) = (Mu)'(0) - Au(0) = f(0) = y. \tag{3.2}$$

Therefore $(i\eta M - A)$ is surjective. By (3.2) we have $u(t) = e^{i\eta t}(i\eta M - A)^{-1}y$. Denote by $e_\eta \otimes x$ the function $t \to (e_\eta \otimes x)(t) := e^{i\eta t}x$. Since $||e_\eta \otimes x||_\alpha = \gamma_\alpha|\eta|^\alpha||x||$, where

$\gamma_\alpha = 2\sup_{t>0} t^{-\alpha}\sin(t/2)$ (see [2, Section 3]), we have

$$\begin{aligned} \gamma_\alpha|\eta|^\alpha||(i\eta M - A)^{-1}y|| &= ||e_\eta \otimes (i\eta M - A)^{-1}y||_\alpha = ||u||_\alpha \\ &\leq ||u||_H \leq ||L||\,||f||_{C^\alpha} \\ &= ||L||(\gamma_\alpha|\eta|^\alpha + 1)||y||. \end{aligned}$$

Hence,

$$||(i\eta M - A)^{-1}y|| \leq ||L||\left(1 + \gamma_\alpha^{-1}|\eta|^{-\alpha}\right)||y||.$$

Thus $(i\eta M - A)^{-1}$ is a bounded operator for every $\eta \in \mathbb{R}\setminus\{0\}$. For $\eta = 0$, observe that by the closed graph theorem A^{-1} is an isomorphism of X onto $D(A)\cap D(M)$ (seen as a Banach space with the graph norm). We conclude that $i\eta \in \rho_M(A)$ for all $\eta \in \mathbb{R}$.

On the other hand, since by the closed graph theorem, $L : C^\alpha(\mathbb{R}; X) \to C^\alpha(\mathbb{R}; [D(A)\cap D(M)]) \cap H^{1,\alpha}(\mathbb{R}; [D(M)])$ is a bounded operator, we have that for all $f \in C^\alpha(\mathbb{R}; X)$ there exist $u \in C^\alpha(\mathbb{R}; [D(A)\cap D(M)]) \cap H^{1,\alpha}(\mathbb{R}; [D(M)])$ and a constant $C > 0$ (independent of f) such that

$$||(Mu)'||_{C^\alpha} + ||Au||_{C^\alpha} \leq C||f||_{C^\alpha}. \tag{3.3}$$

For $f(t) = e^{i\eta t}y$ where $y \in X$ and $\eta \in \mathbb{R}$, the solution u of (3.1) is given by $u(t) = e^{i\eta t}(i\eta M - A)^{-1}y$. Therefore

$$\begin{aligned} ||(Mu)'||_\alpha &= ||e_\eta \otimes i\eta M(i\eta M - A)^{-1}y||_\alpha \\ &= \gamma_\alpha|\eta|^\alpha||i\eta M(i\eta M - A)^{-1}y||. \end{aligned}$$

Since $(Mu)' \in C^\alpha(\mathbb{R}; X)$ and $||(Mu)'||_\alpha \leq C||f||_{C^\alpha} = C(\gamma_\alpha|\eta|^\alpha + 1)||y||$ we have

$$\gamma_\alpha|\eta|^\alpha||i\eta M(i\eta M - A)^{-1}y|| \leq C(\gamma_\alpha|\eta|^\alpha + 1)||y||. \tag{3.4}$$

From (3.4) we have that for $\epsilon > 0$,

$$\sup_{|\eta|>\epsilon}||i\eta M(i\eta M - A)^{-1}|| \leq C\sup_{|\eta|>\epsilon}(1 + \gamma_\alpha^{-1}|\eta|^{-\alpha}) < \infty.$$

By continuity it follows that $\sup_{\eta\in\mathbb{R}}||i\eta M(i\eta M - A)^{-1}|| < \infty$. ■

The following is the main results in this paper. It corresponds to an extension of [2, Theorem 6.1] in case $M = I$.

Theorem 3.4. *Let $A : D(A) \subseteq X \to X$, $M : D(M) \subseteq X \to X$ closed linear operators defined on a Banach space X satisfying $D(A)\cap D(M) \neq \{0\}$. Then, the following assertions are equivalent*

(*i*) *The equation (3.1) is C^α-well posed;*

(*ii*) *$i\mathbb{R} \subset \rho_M(A)$ and $\sup_{\eta\in\mathbb{R}}||i\eta M(i\eta M - A)^{-1}|| < \infty$.*

Proof. (*ii*) $\Rightarrow$ (*i*). For $t \in \mathbb{R}$, define the operator $N(t) := (itM - A)^{-1}$. Observe that by hypothesis $N \in C^2(\mathbb{R}; \mathcal{B}(X, [D(A)\cap D(M)]))$. We claim that N is a $\dot{C}^\alpha$-multiplier. In fact, since $0 \in \rho_M(A)$, we have that A^{-1} is bounded (seen as an operator from X to $[D(A)\cap D(M)]$). The resolvent identity $itMN(t) - I = AN(t)$ implies $A^{-1}(itMN(t) - I) = N(t)$ and therefore, by hypothesis we have that $\sup_{t\in\mathbb{R}}||N(t)|| < \infty$. On the other hand,

$$N'(t) = -iN(t)MN(t),$$

$$N''(t) = -2N(t)MN(t)MN(t).$$

Hence,

$$tN'(t) = -itN(t)MN(t),$$
$$t^2N''(t) = -2N(t)tMN(t)tMN(t).$$

From the hypothesis and the above identities, we have

$$\sup_{t\in\mathbb{R}} ||tN'(t)|| < \infty \quad \text{and} \quad \sup_{t\in\mathbb{R}} ||t^2N''(t)|| < \infty.$$

We conclude from Theorem 2.2 that N is a $\dot{C}^\alpha$-multiplier, with $N \in C^2(\mathbb{R}; \mathcal{B}(X, [D(A) \cap D(M)]))$.

Define $S(t) := (\mathrm{id} \cdot MN)(t)$, where $\mathrm{id}(t) = it$. Observe that by hypothesis $S \in C^2(\mathbb{R}; \mathcal{B}(X))$. Moreover,

$$S'(t) = iMN(t) + tMN(t)MN(t),$$

$$S''(t) = 2MN(t)MN(t) - 2itMN(t)MN(t)MN(t).$$

and

$$tS'(t) = itMN(t) + tMN(t)tMN(t),$$
$$t^2S''(t) = 2tMN(t)tMN(t) - 2itMN(t)tMN(t)tMN(t).$$

Hence, from hypothesis $\sup_{t\in\mathbb{R}} ||S(t)|| < \infty$, $\sup_{t\in\mathbb{R}} ||tS'(t)|| < \infty$ and $\sup_{t\in\mathbb{R}} ||t^2S''(t)|| < \infty$. We conclude that S is a $\dot{C}^\alpha$-multiplier by Theorem 2.2.

Let $f \in C^\alpha(\mathbb{R}; X)$. Since N and S are $\dot{C}^\alpha$-multipliers, there exist $\overline{u} \in C^\alpha(\mathbb{R}; [D(A) \cap D(M)])$, and $v \in C^\alpha(\mathbb{R}; X)$ such that

$$\int_{\mathbb{R}} \overline{u}(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi \cdot N)(s)f(s)ds, \tag{3.5}$$

$$\int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot S)(s)f(s)ds, \tag{3.6}$$

for all $\phi, \varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Let $\phi = \mathrm{id} \cdot \varphi$. From (3.5) we have

$$\int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(\mathrm{id} \cdot \varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\mathrm{id} \cdot \varphi \cdot N)(s)f(s)ds. \tag{3.7}$$

Observe that $\overline{u}(t) \in D(A) \cap D(M)$ and $\mathcal{F}(\phi \cdot N)(s)x \in D(A) \cap D(M)$ for all $x \in X$, $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$. Using the fact that M is closed with $D(A) \cap D(M) \neq \{0\}$, we have from (3.5), (3.6) and (3.7) that

$$\int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id} \cdot \varphi)(s)ds = A\int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(\varphi)(s)ds + \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot I)(s)f(s)ds. \tag{3.8}$$

Moreover, from (3.6) and (3.7) we have

$$\int_{\mathbb{R}} M\overline{u}(s)\mathcal{F}(\mathrm{id} \cdot \varphi)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot S)(s)f(s)ds = \int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds. \tag{3.9}$$

Since $\overline{u} \in C^\alpha(\mathbb{R}; [D(A) \cap D(M)])$ and $D(A) \cap D(M) \neq \{0\}$, we have that $M\overline{u} \in C^\alpha(\mathbb{R}; X)$. It follows from (3.9) and Lemma 2.7 that $(M\overline{u})' = v + y_1$ where $y_1 \in X$. Clearly $v + y_1 \in C^\alpha(\mathbb{R}; X)$, and therefore $\overline{u} \in H^{1,\alpha}(\mathbb{R}; [D(M)])$. From (3.8) and (3.9) we have

$$\int_{\mathbb{R}} v(s)\mathcal{F}(\varphi)(s)ds = A\int_{\mathbb{R}} \overline{u}(s)\mathcal{F}(\varphi)(s)ds + \int_{\mathbb{R}} \mathcal{F}(\varphi \cdot I)(s)f(s)ds.$$

From Lemma 2.6 we obtain $v = A\overline{u} + f + y_2$ where $y_2 \in X$. Therefore $(M\overline{u})' = A\overline{u} + f + y_3$ with $y_3 = y_1 + y_2$. Let $u(t) = \overline{u}(t) + x$ where $x = A^{-1}y_3$. Note that x is well defined since $i\mathbb{R} \subset \rho_M(A)$. Since $\overline{u} \in C^\alpha(\mathbb{R}; [D(A) \cap D(M)]) \cap H^{1,\alpha}(\mathbb{R}; [D(M)])$ we have $u \in C^\alpha(\mathbb{R}; [D(A) \cap D(M)]) \cap H^{1,\alpha}(\mathbb{R}; [D(M)])$. An easy computation shows that u satisfies the equation (3.1). To see the uniqueness, suppose that

$$(Mu)'(t) = Au(t), \quad t \in \mathbb{R}, \tag{3.10}$$

where $u \in C^\alpha(\mathbb{R}; [D(A) \cap D(M)])$ with $u \in H^{1,\alpha}(\mathbb{R}; [D(M)])$. Since M is a closed operator, we have $\widehat{Mu'}(\lambda) = \lambda M\hat{u}(\lambda) - Mu(0)$ $(\mathrm{Re}\lambda \neq 0)$. Since $\hat{u}(\lambda) \in D(A) \cap D(M) \neq \{0\}$ we obtain, $(\lambda M - A)\hat{u}(\lambda) = Mu(0)$ for all $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. Since $i\mathbb{R} \subset \rho_M(A)$ it follows that the Carleman spectrum $\mathrm{sp}_C(u)$ of u is empty. Therefore $u \equiv 0$ (see [3, Theorem 4.8.2]). We conclude that the problem (3.1) is C^α-well posed.

$(i) \Rightarrow (ii)$. Follows from Proposition 3.3. ∎

Corollary 3.5. *In the context of Theorem 3.4, if condition (ii) is fulfilled, we have* $(Mu)', Au \in C^\alpha(\mathbb{R}; X)$. *Moreover, there exists a constant* $C > 0$ *independent of* $f \in C^\alpha(\mathbb{R}; X)$ *such that*

$$||(Mu)'||_{C^\alpha(\mathbb{R};X)} + ||Au||_{C^\alpha(\mathbb{R};X)} \leq C||f||_{C^\alpha(\mathbb{R};X)}. \tag{3.11}$$

Remark 3.6.

The inequality (3.11) is a consequence of the Closed Graph Theorem and known as the *maximal regularity property* for equation (3.1). We deduce that the operator S defined by:

$$(Su)(t) = (Mu)'(t) - Au(t)$$

with domain

$$D(S) = H^{1,\alpha}(\mathbb{R}; [D(M)]) \cap C^\alpha(\mathbb{R}; [D(A) \cap D(M)]),$$

is an isomorphism onto. In fact, by Remark 3.2 we have that the space $H := C^\alpha(\mathbb{R}; [D(A) \cap D(M)]) \cap H^{1,\alpha}(\mathbb{R}; [D(M)])$ becomes a Banach space under the norm

$$||u||_H := ||u||_{C^\alpha(\mathbb{R};X)} + ||(Mu)'||_{C^\alpha(\mathbb{R};X)} + ||Au||_{C^\alpha(\mathbb{R};X)}.$$

We remark that such isomorphisms are crucial for the handling of nonlinear evolution equations (see [1]). Indeed, assume that X is a Banach space and A, M satisfy the condition (ii) in Theorem 3.4. Consider the semilinear problem

$$(Mu)'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}. \tag{3.12}$$

Define the Nemytskii's superposition operator $N : H \to C^\alpha(\mathbb{R}; X)$ given by $N(v)(t) = f(t, v(t))$ and the bounded linear operator

$$T := S^{-1} : C^\alpha(\mathbb{R}; X) \to H$$

by $T(g) = u$ where u is the unique solution to linear problem

$$(Mu)'(t) = Au(t) + g(t), \quad t \geq 0.$$

Then, to solve (3.12) we need to show that the operator $R : H \to H$ defined by $R = TN$ has a fixed point. For more details, we refer to H. Amann [1], H. Brill [7] and A. Rutkas and L. Vlasenko [24].

4. Examples

Example 4.7.

Let us consider the problem

$$\frac{\partial(m(x)u)}{\partial t} - \Delta u = f(t,x), \text{ in } \mathbb{R} \times \Omega \tag{4.13}$$

$$u = 0, \text{ in } \mathbb{R} \times \partial\Omega, \tag{4.14}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m(x) \geq 0$ is a given measurable bounded function on Ω and f is a function on $\mathbb{R} \times \Omega$.

Let M be the multiplication operator by m. If we take $X = H^{-1}(\Omega)$ then by [6, p.38] (see also references therein), we have that there exists a constant $c > 0$ such that

$$||M(zM - \Delta)^{-1}|| \leq \frac{c}{1 + |z|},$$

whenever $\text{Re}z \geq -c(1 + |\text{Im}(z)|)$. In particular, in the imaginary axis we have $||M(itM - \Delta)^{-1}|| \leq \frac{c}{1+|t|}$, for all $t \in \mathbb{R}$. Therefore, we conclude by Theorem 3.4 that the equation (4.13) is C^α-well posed. Thus, given $f \in C^\alpha(\mathbb{R} \times \Omega; X)$ there exists a unique solution u to problem (4.13) which satisfy $(m(x)u)', \Delta u \in C^\alpha(\mathbb{R} \times \Omega; X)$.

Example 4.8.

Let P be a densely defined positive selfadjoint operator defined on a Hilbert space X with $P \geq \delta > 0$. Let $M = P - \varepsilon$ with $\varepsilon \leq \delta$, and let $A = -\sum_{i=0}^{k} a_i P^i$ with $a_i \geq 0$, $a_k > 0$, and $k \geq 2$ is an integer. From [14, p. 73] we have that there exists a constant $c > 0$ such that

$$||M(zM - A)^{-1}|| \leq \frac{c}{1 + |z|},$$

whenever $\text{Re}z \geq -c(1+|\text{Im}(z)|)$. Thus, in the imaginary axis we have $||M(itM - A)^{-1}|| \leq \frac{c}{1+|t|}$, for all $t \in \mathbb{R}$. Hence, in this conditions the equation (3.1) is C^α-well posed.

Example 4.9.

For $(x,t) \in \Omega \times \mathbb{R}$ where $\Omega = (0,1)$, consider the problem

$$\frac{\partial}{\partial t}\Big(1 - \frac{\partial^2}{\partial x^2}\Big)u(x,t) = -\frac{\partial^4}{\partial x^4}u(x,t) + f(x,t) \tag{4.15}$$

$$u = 0, \quad \text{in } \partial\Omega \times \mathbb{R}. \tag{4.16}$$

In the space $X = L^2(\Omega)$, let $P = -\frac{\partial^2}{\partial x^2}$, with domain $D(P) = H^2(\Omega) \cap H_0^1(\Omega)$. Observe that P is a positive selfadjoint operator in X. If $M = P + I$, and $A = -P^2$, then the equation (4.15) can be written in the form of (3.1). By Example 4.8, the equation (4.15) is C^α-well posed.

Example 4.10.

Consider the problem

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2}{\partial x^2} + 1\Big)u(t,x) = -a\frac{\partial^2}{\partial x^2}u(t,x) - ku(t,x) + f(t,x), \quad t \in \mathbb{R}, x \in [0,\pi] \tag{4.17}$$

$$u(t,0) = u(t,\pi) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,\pi) = 0, \quad t \in \mathbb{R} \tag{4.18}$$

where a is positive constant and $-2a < k < 4a$. In $X = C_0([0,\pi]) = \{u \in C([0,\pi]) : u(0) = u(\pi)\}$ take K the realization of $\frac{\partial^2}{\partial x^2}$ with domain

$$D(K) = \{u \in C^2([0,\pi]) : u(0) = u(\pi) = \frac{\partial^2}{\partial x^2}u(0) = \frac{\partial^2}{\partial x^2}u(\pi) = 0\}.$$

If $M = K + I$, and $A = aM + (k-a)I$, then the equation (4.17) can be written in the form of (3.1). By [6, p.39] or [14] we have, as in the above example:

$$||M(itM - A)^{-1}|| \leq \frac{c}{1+|t|}$$

for all $t \in \mathbb{R}$. Therefore, by Theorem 3.4 the equation (4.17) is C^α-well posed, that is, for all $f \in C^\alpha(\mathbb{R} \times [0,\pi]; C_0([0,\pi]))$ there exists a unique solution u of (4.17) with maximal regularity $\frac{\partial^2 u}{\partial x^2} \in C^\alpha(\mathbb{R} \times [0,\pi]; C_0([0,\pi]))$.

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Universidad de Talca, Instituto de Matemática y Física, Casilla 747, Talca-Chile.
E-mail address: rponce@inst-mat.utalca.cl