BOUNDED MILD SOLUTIONS TO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES.

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ABSTRACT. We study the existence and uniqueness of bounded solutions for the semilinear fractional differential equation

$$D^{\alpha}u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t,u(t)) \quad t \in \mathbb{R},$$

where A is a closed linear operator defined on a Banach space X, $\alpha > 0$, $a \in L^1(\mathbb{R}_+)$ is a scalar-valued kernel and $f : \mathbb{R} \times X \to X$ satisfying Lipschitz type conditions. Sufficient conditions are established for the existence and uniqueness of an almost periodic, almost automorphic and asymptotically almost periodic solution, among other.

1. INTRODUCTION

Fractional differential equations have been used by many researchers to adequately describe the evolution of a variety of physical and biological processes. Examples include the nonlinear oscillation of earthquake, electrochemistry, electromagnetism, viscoelasticity and rheology. See, for instance, [3, 18, 19] and [27] for more details.

In this paper, we consider the following semilinear fractional differential equation with infinite delay

$$D^{\alpha}u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t,u(t)), \qquad t \in \mathbb{R}, \tag{1.1}$$

where A is a closed linear operator defined on a Banach space X, $a \in L^1(\mathbb{R}_+)$ is a scalar-valued kernel, f belongs to a closed subspace of the space of continuous and bounded functions, and for $\alpha > 0$, the fractional derivative is understood in the Weyl's sense.

Under appropriate assumptions on A and f, we want to prove that (1.1) has a unique *mild* solution u which behaves in the same way that f. For example, we want to find conditions implying that u is almost periodic (resp. automorphic) if $f(\cdot, x)$ is almost periodic (resp. almost automorphic).

When $\alpha = 1$ in equation (1.1), we obtain the equation with infinite delay

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t,u(t)), \qquad t \in \mathbb{R}. \tag{1.2}$$

Equations of this kind arise, for example, in the study of heat flow in materials of fading memory type as well as some equations of population dynamics or in viscoelasticity. See [35] and [36,

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Chapter III, Section 13] for more details. Existence of almost periodic or almost automorphic (among other) mild solutions to equation (1.2) has been recently studied in [24, 26].

A characterization of the existence and uniqueness of periodic (strong) solutions to the linear problem (1.1), that is, when $f(t, u(t)) = g(t)$, on periodic vector-valued Lebesgue spaces and in the scale of periodic Besov spaces, have been studied in [9] using Fourier multipliers.

When $a \equiv 0$, sufficient conditions for the existence and uniqueness of mild solutions in the cases $\alpha = 1$, $\alpha = 2$, with f almost periodic or almost automorphic, among other, have been obtained by several authors in [5, 6, 14, 15, 16, 32] for the case $\alpha = 1$, and in [5, 21, 34] for $\alpha = 2$.

The fractional case, $\alpha > 0$,

$$D_t^{\alpha} u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{1.3}$$

where the fractional derivative is taken in the Riemann-Liouville's sense, have been studied in several papers. See [2, 4, 11, 13, 30, 37] and the references therein. See moreover [1, Section 3].

In this paper, we study the existence and uniqueness of mild solutions for (1.1) where the input data f belongs to some of above functions spaces. Concretely, we prove that if f is for example almost periodic (resp. almost automorphic) and satisfies some Lipschitz type conditions, then there exists a unique mild solution u of (1.1) which is almost periodic (resp. almost automorphic) and is given by

$$u(t) = \int_{-\infty}^{t} S_{\alpha}(t-s) f(s, u(s)) ds, \quad t \in \mathbb{R}, \tag{1.4}$$

where $\{S_{\alpha}(t)\}_{t \geq 0}$ is the α-*resolvent family* generated by A. It is remarkable that, in the scalar case, that is $A = -\rho I$, with $\rho > 0$, some concrete examples of integrable α-resolvent families are showed. See Examples 4.17 and 4.18 below.

The paper is organized as follows. In the second section, we review recent results about several intermediate Banach spaces interpolating between periodic and bounded continuous functions. In Section 3, we study the linear case of equation (1.1), that is, when $f(t, u(t)) = g(t)$ for all $t \in \mathbb{R}$. Assuming that A generates an α-resolvent family we ensure the existence and uniqueness of mild solutions in each class of function spaces introduced in section 2. Some properties of the Mittag-Leffler functions are also studied. Section 4 is devoted to the semilinear case. There, using the previous results on the linear case and the Banach contraction principle we give sufficient conditions that ensure the existence and uniqueness of (mild) solutions almost periodic, almost automorphic, among other, for equation (1.1). Some examples completes this section.

2. Preliminaries

For a complex Banach space $(X, ||\cdot||)$, we denote

$$BC(X) := \{f : \mathbb{R} \to X; f \text{ is continuous }, ||f||_{\infty} := \sup_{t \in \mathbb{R}} ||f(t)|| < \infty\}.$$

Let $P_{\omega}(X) := \{f \in BC(X) : f \text{ is continuous } : \exists \omega > 0, f(t+\omega) = f(t), \text{ for all } t \in \mathbb{R}\}$ be the space of all vector-valued periodic functions. We recall that a function $f \in BC(X)$ is said to be almost periodic (in the sense of Bohr) if for any $\varepsilon > 0$, there exists $\omega = \omega(\varepsilon) > 0$ such that every subinterval $\mathbb{R}$ of length ω contains at least one point τ such that $||f(t+\tau) - f(t)||_{\infty} < \varepsilon$. We denote by $AP(X)$ the set of all these functions. The space of compact almost automorphic functions will be denoted by $AA_c(X)$. Recall that function $f \in BC(X)$ belongs to $AA_c(X)$ if and only if for all sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t+s_n)$ and $f(t) = \lim_{n\to\infty} g(t-s_n)$ uniformly over compact subsets of $\mathbb{R}$. Clearly the function g above is continuous on $\mathbb{R}$. Finally, a function $f \in BC(X)$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t-s_n), \qquad \text{for each } t \in \mathbb{R}.$$

Almost automorphicity, as a generalization of the classical concept of an almost periodic function, was introduced in the literature by S. Bochner and recently studied by several authors, including [8, 10, 12, 16, 21, 32] among others. A complete description of their properties and further applications to evolution equations can be found in the monographs [31] and [33] by G. M. N'Guérékata.

We recall that $AA_c(X)$ and $AA(X)$ are Banach spaces under the norm $||\cdot||_\infty$ and

$$P_\omega(X) \subset AP(X) \subset AA_c(X) \subset AA(X) \subset BC(X).$$

Now we consider the set $C_0(X) := \{f \in BC(X) : \lim_{|t|\to\infty} ||f(t)|| = 0\}$, and define the space of asymptotically periodic functions as $AP_\omega(X) := P_\omega(X) \oplus C_0(X)$. Analogously, we define the space of asymptotically almost periodic functions,

$$AAP(X) := AP(X) \oplus C_0(X),$$

the space of asymptotically compact almost automorphic functions,

$$AAA_c(X) := AA_c(X) \oplus C_0(X),$$

and the space of asymptotically almost automorphic functions,

$$AAA(X) := AA(X) \oplus C_0(X).$$

We have the following natural proper inclusions

$$AP_\omega(X) \subset AAP(X) \subset AAA_c(X) \subset AAA(X) \subset BC(X).$$

Denote by $SAP_\omega(X) := \{f \in BC(X) : \exists \omega > 0, ||f(t+\omega) - f(t)|| \to 0 \text{ as } t \to \infty\}$. The class of functions in $SAP_\omega(X)$ is called S-asymptotically ω-periodic. Now, we consider the following set

$$P_0(X) := \{f \in BC(X) : \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} ||f(s)||ds = 0\},$$

and define the following classes of spaces: the space of pseudo-periodic functions

$$PP_\omega(X) := P_\omega(X) \oplus P_0(X),$$

the space of pseudo-almost periodic functions

$$PAP(X) := AP(X) \oplus P_0(X),$$

the space of pseudo-compact almost automorphic functions

$$PAA_c(X) := AA_c(X) \oplus P_0(X),$$

and the space of pseudo-almost automorphic functions

$$PAA(X) := AA(X) \oplus P_0(X).$$

As before, we also have the following relationship between them;

$$PP_\omega(X) \subset PAP(X) \subset PAA_c(X) \subset PAA(X) \subset BC(X).$$

Denote by $\mathcal{N}(\mathbb{R}, X)$ or simply $\mathcal{N}(X)$ the following function spaces

$$\begin{aligned} \mathcal{N}(X) : &= \{P_\omega(X), AP(X), AA_c(X), AA(X), AP_\omega(X), AAP(X), AAA_c(X), AAA(X), \\ & PP_\omega(X), PAP(X), PAA_c(X), PAA(X), SAP_\omega(X), BC(X)\}. \end{aligned}$$

We recall that a strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is say to be uniformly integrable if

$$||S|| := \int_0^\infty ||S(t)||dt < \infty.$$

The following Theorem is taken from [24].

Theorem 2.1 ([24]). *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a uniformly integrable and strongly continuous family. If f belongs to one of the spaces of $\mathcal{N}(X)$, then*

$$\int_{-\infty}^t S(t-s)f(s)ds,$$

belongs to the same space as f.

We define the set $\mathcal{N}(\mathbb{R} \times X; X)$ which consists of all continuous functions $f : \mathbb{R} \times X \to X$ such that $f(\cdot, x) \in \mathcal{N}(\mathbb{R}, X)$ uniformly for each $x \in K$, where K is any bounded subset of X.

We recall from [24] that $\mathcal{M}(\mathbb{R}, X)$, or simply $\mathcal{M}(X)$, denotes one of the spaces $P_\omega(X)$, $AP_\omega(X), PP_\omega(X)$, $SAP_\omega(X), AP(X), AAP(X), PAP(X), AA(X), AAA(X)$ and $PAA(X)$. Define the set $\mathcal{M}(\mathbb{R}\times X, X)$ of all continuous functions $f : \mathbb{R}\times X \to X$ such that $f(\cdot, x) \in \mathcal{M}(\mathbb{R}, X)$ uniformly for each $x \in K$, where K is any bounded subset of X. We have the following composition theorem.

Theorem 2.2 ([24]). *Let $f \in \mathcal{M}(\mathbb{R} \times X, X)$ be given and assume that there exists a constant L_f such that*

$$||f(t, u) - f(t, v)|| \leq L_f||u - v||,$$

for all $t \in \mathbb{R}$ and $u, v \in X$. If $\psi \in \mathcal{M}(X)$, then $f(\cdot, \psi(\cdot)) \in \mathcal{M}(X)$.

Given a function $g : \mathbb{R} \to X$, the *Weyl fractional integral* of order $\alpha > 0$ is defined by

$$D^{-\alpha}g(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-s)^{\alpha-1}g(s)ds, \quad t \in \mathbb{R},$$

when this integral is convergent. The *Weyl fractional derivatives* $D^\alpha g$ of order $\alpha > 0$ is defined by

$$D^\alpha g(t) := \frac{d^n}{dt^n} D^{-(n-\alpha)}g(t), \quad t \in \mathbb{R},$$

where $n = [\alpha]+1$. It is known that $D^\alpha D^{-\alpha}g = g$ for any $\alpha > 0$, and $D^n = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$. See [28] and [29] for more details.

The Mittag-Leffler function (see e.g. [27]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{Ha} e^\mu \frac{\mu^{\alpha-\beta}}{\mu^\alpha - z} d\mu, \quad \alpha, \beta > 0, z \in \mathbb{C},$$

where Ha is a Hankel path, i.e. a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ counterclockwise. The Laplace transform of the Mittag-Leffler function is given by ([17, pp. 267]):

$$\mathcal{L}(t^{\beta-1}E_{\alpha,\beta}(-\rho t^{\alpha}))(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}+\rho}, \quad \rho \in \mathbb{C}, \mathrm{Re}\lambda > |\rho|^{1/\alpha}.$$

Definition 2.3. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X, and $\alpha > 0$. Given $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, we say that A is the generator of an α-resolvent family, if there exists $\omega \geq 0$ and a strongly continuous function $S_\alpha : [0,\infty) \to \mathcal{B}(X)$ such that $\{\frac{\lambda^\alpha}{1+\hat{a}(\lambda)} : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$ and for all $x \in X$,*

$$\big(\lambda^\alpha - (1+\hat{a}(\lambda))A\big)^{-1}x = \frac{1}{1+\hat{a}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{a}(\lambda)} - A\right)^{-1}x = \int_0^\infty e^{-\lambda t}S_\alpha(t)x\,dt, \;\; \mathrm{Re}\lambda > \omega.$$

In this case, $\{S_\alpha(t)\}_{t\geq 0}$ is called the α-resolvent family *generated by A.*

Remark 2.4.

Observe that if $b(t) = g_\alpha(t) + (g_\alpha * a)(t)$, $(t \geq 0)$ where $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $(g_\alpha * a)(t) = \int_0^t g_\alpha(t-s)a(s)ds$, then, we have that the family α-resolvent $\{S_\alpha(t)\}_{t\geq 0}$ is an (b, g_α)-regularized family. In particular, if $a \equiv 0$, a 1-resolvent family is the same as a C_0-semigroup, whereas that a 2-resolvent family corresponds to the concept of sine family. See [22, 23] and [25]. Therefore, if A is the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ then by [22, Proposition 3.1 and Lemma 2.2] we have that, $\{S_\alpha(t)\}_{t\geq 0}$ verify the following properties:

i) $S_\alpha(0) = g_\alpha(0)$;
ii) $S_\alpha(t)x \in D(A)$ and $S_\alpha(t)Ax = AS_\alpha(t)x$ for all $x \in D(A)$ and $t \geq 0$;
iii) $S_\alpha(t)x = g_\alpha(t)x + \int_0^t b(t-s)AS_\alpha(s)xds$, for all $x \in D(A)$ and $t \geq 0$;
iv) $\int_0^t b(t-s)S_\alpha(s)xds \in D(A)$ and $S_\alpha(t)x = g_\alpha(t)x + A\int_0^t b(t-s)S_\alpha(s)xds$, for all $x \in X$ and $t \geq 0$.

Sufficient conditions for $\{S_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ to be a resolvent family can be obtained from [22, Theorem 3.4] and [25, Theorems 4.1,4.3].

3. Bounded mild solutions to the linear case

Consider the following linear equation

$$D^\alpha u(t) = Au(t) + (a\dot{*}Au)(t) + f(t), \quad t \in \mathbb{R}, \tag{3.5}$$

where $(a\dot{*}Au)(t) = \int_{-\infty}^t a(t-s)Au(s)ds$. Assume that A is the generator of an α-resolvent family $\{S(t)\}_{t\geq 0}$ which is uniformly integrable. Given $f \in \mathcal{N}(X)$, let $\phi(t)$ be the function given by

$$\phi(t) := \int_{-\infty}^t S_\alpha(t-s)f(s)ds, \quad t \in \mathbb{R}. \tag{3.6}$$

Then, $||\phi||_\infty \leq ||S_\alpha||\,||f||_\infty$. If $f(t) \in D(A)$ for all $t \in \mathbb{R}$, then $\phi(t) \in D(A)$ for all $t \in \mathbb{R}$ (see [7, Proposition 1.1.7]). Since $\{S_\alpha(t)\}_{t\geq 0}$ is integrable, $\phi \in \mathcal{N}(X)$ by Theorem 2.1. Assume that $D^\alpha\phi$ exists. Let $b(t) = g_\alpha(t) + (g_\alpha * a)(t)$, and $n = [\alpha]+1$. We obtain by Remark 2.4 and from

Fubini's theorem that

$$\begin{aligned}
D^{\alpha}\phi(t) &= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\phi(s)ds \\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s} S_{\alpha}(s-r)f(r)drds \\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\Big[g_{\alpha}(s-r)f(r)+(b*AS_{\alpha})(s-r)f(r)\Big]drds \\
&= \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)D^{-\alpha}f(s)ds + \\
&\qquad \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\int_{0}^{s-r} b(s-r-v)AS_{\alpha}(v)f(r)dvdrds \\
&= f(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}[g_{\alpha}(s-w)+ \\
&\qquad (g_{\alpha}*a)(s-w)]\int_{-\infty}^{w} AS_{\alpha}(w-r)f(r)drdwds \\
&= f(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s} g_{\alpha}(s-w)A\phi(w)dwds + \\
&\qquad \frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}(g_{\alpha}*a)(s-w)A\phi(w)dwds \\
&= f(t)+A\phi(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s}\int_{-\infty}^{v} g_{\alpha}(s-v)a(v-w)A\phi(w)dwdvds \\
&= f(t)+A\phi(t)+\frac{d^n}{dt^n}\int_{-\infty}^{t} g_{n-\alpha}(t-s)\int_{-\infty}^{s} g_{\alpha}(s-v)(a\dot{*}A\phi)(v)dvds \\
&= f(t)+A\phi(t)+(a\dot{*}A\phi)(t).
\end{aligned}$$

That is, ϕ is a (strict) solution of Eq. (3.5). Since in general, we have only $f(t)\in X$ or that $D^{\alpha}\phi$ does not exists, in what follows, we will say that $\phi(t)$ defined by (3.6) is a *mild solution* of Eq. (3.5).

Theorem 3.5. *Assume that A generates an α-resolvent family $\{S_{\alpha}(t)\}_{t\geq 0}$ such that*

$$||S_{\alpha}(t)|| \leq \phi_{\alpha}(t), \quad \text{for all } t\geq 0 \quad \text{with } \phi_{\alpha}\in L^1(\mathbb{R}_+).$$

If $f\in\mathcal{N}(X)$, then the equation (3.5) has a unique mild solution $u\in\mathcal{N}(X)$.

Proof. If $f\in\mathcal{N}(X)$, then u defined by $u(t):=\int_{-\infty}^{t} S_{\alpha}(t-s)f(s)ds$ is well defined and by Theorem 2.1, $u\in\mathcal{N}(X)$. Therefore, u is the unique mild solution of (3.5). ☐

In what follows, we denote $e_{\alpha,\beta}(t)=t^{\beta-1}E_{\alpha,\beta}(-\rho t^{\alpha}), \rho\in\mathbb{R}$. It is known that ([17, pp. 268]) for $0<\alpha\leq\beta<1$,

$$e_{\alpha,\beta}(t)=\frac{1}{\pi}\int_{0}^{\infty} e^{-rt}K_{\alpha,\beta}(r)dr, \quad t\geq 0, \tag{3.7}$$

where

$$K_{\alpha,\beta}(r) = \frac{[r^\alpha \sin(\beta\pi) + \rho \sin((\beta-\alpha)\pi)]}{r^{2\alpha} + 2\rho r^\alpha \cos(\alpha\pi) + \rho^2} r^{\alpha-\beta}. \tag{3.8}$$

The following result shows that a similar representation to (3.7) of $e_{\alpha,\beta}$ holds if $1 < \beta \leq \alpha < 2$.

Proposition 3.6. *Let $1 < \beta \leq \alpha < 2$, and $\rho \in \mathbb{R}$. For all $t \geq 0$ we have:*

$$e_{\alpha,\beta}(t) = \frac{1}{\pi}\int_0^\infty e^{-rt}K_{\alpha,\beta}(r)dr + \frac{2}{\alpha}\rho^{(1-\beta)/\alpha}e^{\rho^{1/\alpha}t\cos(\pi/\alpha)}\cos\left(\rho^{1/\alpha}t\sin(\pi/\alpha) + \frac{(1-\beta)\pi}{\alpha}\right), \tag{3.9}$$

where, $K_{\alpha,\beta}$ is defined by (3.8).

Proof. Follows the same lines of [4] and [17]. From the inversion complex formula for the Laplace transform, we have

$$e_{\alpha,\beta}(t) = \frac{1}{2\pi i}\int_{B_r} e^{\lambda t}\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha + \rho}d\lambda,$$

where B_r denotes the Bromwich path, i.e. a line $\mathrm{Re}(\lambda) = \sigma \geq \rho^{1/\alpha}$ and $\mathrm{Im}(\lambda)$ running from $-\infty$ to ∞. As in [17] we obtain a decomposition of $e_{\alpha,\beta}$ in two parts,

$$e_{\alpha,\beta}(t) = f_{\alpha,\beta}(t) + g_{\alpha,\beta}(t),$$

where by a Titchmarsch's formula (see [27, pp. 225])

$$\begin{aligned} f_{\alpha,\beta}(t) &= \frac{1}{\pi}\int_0^\infty e^{-rt}\mathrm{Im}\left(\left.\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha+\rho}\right|_{\lambda=re^{i\pi}}\right)dr \\ &= \frac{1}{\pi}\int_0^\infty e^{-rt}\frac{[r^\alpha \sin(\beta\pi) + \rho\sin((\beta-\alpha)\pi)]}{r^{2\alpha}+2\rho r^\alpha\cos(\alpha\pi)+\rho^2}r^{\alpha-\beta}dr \end{aligned}$$

and

$$g_{\alpha,\beta}(t) = e^{s_0 t}\mathrm{Res}\left(\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha+\rho}\right)\Bigg|_{s_0} + e^{s_1 t}\mathrm{Res}\left(\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha+\rho}\right)\Bigg|_{s_1} = \frac{1}{\alpha}(e^{s_0 t}s_0^{1-\beta} + e^{s_1 t}s_1^{1-\beta}),$$

where $s_0 = \rho^{1/\alpha}e^{i\pi/\alpha}$ and $s_1 = \rho^{1/\alpha}e^{-i\pi/\alpha}$ are the poles of $\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha+\rho}$ $(1 < \beta \leq \alpha < 2)$. Therefore

$$g_{\alpha,\beta}(t) = \frac{2}{\alpha}\rho^{(1-\beta)/\alpha}e^{\rho^{1/\alpha}t\cos(\pi/\alpha)}\cos\left(\rho^{1/\alpha}t\sin(\pi/\alpha) + \frac{(1-\beta)\pi}{\alpha}\right). \tag{3.10}$$

□

Remark 3.7. *We notice that if $0 < \beta \leq \alpha < 1$, then $\frac{\lambda^{\alpha-\beta}}{\lambda^\alpha+\rho}$ has no poles, and as consequence $g_{\alpha,\beta}(t) = 0$, $t \geq 0$. Hence, if $0 < \beta \leq \alpha < 1$, then*

$$e_{\alpha,\beta}(t) = f_{\alpha,\beta}(t) = \frac{1}{\pi}\int_0^\infty e^{-rt}K_{\alpha,\beta}(r)dr.$$

Remark 3.8. *Since for $1 < \beta \leq \alpha < 2$, $0 = e_{\alpha,\beta}(0) = f_{\alpha,\beta}(0) + g_{\alpha,\beta}(0)$ we obtain from (3.9) and (3.10) that*

$$\frac{1}{\pi}\int_0^\infty K_{\alpha,\beta}(r)dr = -\frac{2}{\alpha}\rho^{(1-\beta)/\alpha}\cos\left(\frac{(1-\beta)\pi}{\alpha}\right). \tag{3.11}$$

Lemma 3.9. *If $0 < \beta \leq \alpha < 1$ and $\rho > 0$, then $e_{\alpha,\beta} \in L^1(\mathbb{R}_+)$.*

Proof. By Remark 3.7, $g_{\alpha,\beta}(t)=0$ for all $t\geq 0$. First, we prove the Lemma in the case $0<\beta<\alpha<1$. An easy computation using complex analysis (see [20, pp. 199]) shows that if $0<|\gamma|<1$ and $0<|\theta|<\pi$ then

$$\int_0^\infty \frac{x^\gamma}{x^2+2xa\cos\theta+a^2}dx = a^{\gamma-1}\frac{\pi}{\sin\gamma\pi}\frac{\sin\gamma\theta}{\sin\theta}, \quad a>0. \tag{3.12}$$

Applying Fubini's theorem and noting that $K_{\alpha,\beta}(r)=rK_{\alpha,\beta+1}(r)$, we get

$$\begin{aligned}
\int_0^\infty |f_{\alpha,\beta}(t)|dt &\leq \frac{1}{\pi}\int_0^\infty\int_0^\infty e^{-rt}|K_{\alpha,\beta}(r)|drdt\\
&= \frac{1}{\pi}\int_0^\infty r^{-1}|K_{\alpha,\beta}(r)|dr\\
&= \frac{1}{\pi}\int_0^\infty |K_{\alpha,\beta+1}(r)|dr\\
&= \frac{1}{\pi}\int_0^\infty \frac{|r^\alpha\sin((\beta+1)\pi)+\rho\sin(((\beta+1)-\alpha)\pi)|}{r^{2\alpha}+2\rho r^\alpha\cos(\alpha\pi)+\rho^2}r^{\alpha-(\beta+1)}dr\\
&\leq \frac{1}{\pi}\int_0^\infty \frac{r^{2\alpha-(\beta+1)}}{r^{2\alpha}+2\rho r^\alpha\cos(\alpha\pi)+\rho^2}dr+\frac{\rho}{\pi}\int_0^\infty\frac{r^{\alpha-(\beta+1)}}{r^{2\alpha}+2\rho r^\alpha\cos(\alpha\pi)+\rho^2}dr\\
&:= I_1+\rho I_2.
\end{aligned}$$

With the change of variable $x=r^\alpha$, we obtain by (3.12) that

$$I_1 = \frac{1}{\alpha\pi}\int_0^\infty \frac{x^{1-\beta/\alpha}}{x^2+2x\rho\cos(\alpha\pi)+\rho^2}dx=\frac{1}{\alpha}\rho^{-\beta/\alpha}\frac{\sin((\alpha-\beta)\pi)}{\sin\alpha\pi\sin\frac{\beta}{\alpha}\pi}$$

and

$$I_2 = \frac{1}{\alpha\pi}\int_0^\infty \frac{x^{-\beta/\alpha}}{x^2+2x\rho\cos(\alpha\pi)+\rho^2}dx=\frac{1}{\alpha}\rho^{-(1+\beta/\alpha)}\frac{\sin\beta\pi}{\sin\alpha\pi\sin\frac{\beta}{\alpha}\pi},$$

since $-1<1-\beta/\alpha<1$ and $-1<-\beta/\alpha<1$ when $0<\beta<\alpha<1$. Hence,

$$\int_0^\infty |f_{\alpha,\beta}(t)|dt \leq \frac{1}{\alpha}\rho^{-\beta/\alpha}\frac{\sin((\alpha-\beta)\pi)}{\sin\alpha\pi\sin\frac{\beta}{\alpha}\pi}+\frac{1}{\alpha}\rho^{-\beta/\alpha}\frac{\sin\beta\pi}{\sin\alpha\pi\sin\frac{\beta}{\alpha}\pi}<\infty.$$

For $0<\beta=\alpha<1$, we have

$$\begin{aligned}
\int_0^\infty |f_{\alpha,\alpha}(t)|dt &\leq \frac{1}{\pi}\int_0^\infty |K_{\alpha,\alpha+1}(r)|dr\\
&= \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty\frac{r^{\alpha-1}}{r^{2\alpha}+2\rho r^\alpha\cos(\alpha\pi)+\rho^2}dr\\
&= \frac{\sin(\alpha\pi)}{\alpha\pi}\rho^{-1}\int_0^\infty\frac{1}{x^2+2x\cos(\alpha\pi)+1}dx\\
&= \frac{\rho^{-1}}{\alpha\pi}\left(\frac{\pi}{2}-\arctan(\cot(\alpha\pi))\right)<\infty.
\end{aligned}$$

□

Lemma 3.10. *If $1<\beta\leq\alpha<2$ and $\rho>0$, then $e_{\alpha,\beta}\in L^1(\mathbb{R}_+)$.*

Proof. The case $\alpha = \beta$ is contained in the proof of [4, Corollary 3.7]. We prove here the lemma for $1 < \beta < \alpha < 2$. From Proposition 3.6, $e_{\alpha,\beta}(t) = f_{\alpha,\beta}(t) + g_{\alpha,\beta}(t)$. Since

$$|g_{\alpha,\beta}(t)| \leq \frac{2}{\alpha}\rho^{(1-\beta)/\alpha}e^{\rho^{1/\alpha}t\cos(\pi/\alpha)}$$

and $\cos(\pi/\alpha) < 0$ for $1 < \alpha < 2$, we obtain that

$$\int_0^\infty |g_{\alpha,\beta}(t)|dt < \infty.$$

Since for $1 < \beta < \alpha < 2$, $0 < 1 - \beta/\alpha < 1/2$ and $-1 < -\beta/\alpha < -1/2$, we have that $f_{\alpha,\beta} \in L^1(\mathbb{R}_+)$ by (3.12) as in Lemma 3.9. ☐

The following Corollary is a direct consequence of Theorem 3.5, Lemmas 3.9 and 3.10. The case $\alpha = 1$ is proved in [24, Corollary 3.6].

Corollary 3.11. *Let $f \in \mathcal{N}(X)$ and let $\rho > 0$ be a real number. Then, for all $0 < \alpha < 2$ the equation*

$$D^\alpha u(t) = -\rho u(t) + f(t), \quad t \in \mathbb{R},$$

has a unique mild solution u which belongs to the same space as that of f and is given by

$$u(t) = \int_{-\infty}^t S_{\alpha,\alpha}(t-s)f(s)ds,$$

where, for $t \geq 0$,

$$S_{\alpha,\alpha}(t) = \frac{1}{\pi}\sin\pi\alpha\int_0^\infty e^{-rt}\frac{r^\alpha}{r^{2\alpha}+2r^\alpha\rho\cos\pi\alpha+\rho^2}dr, \qquad \text{if } 0 < \alpha < 1, \tag{3.13}$$

and

$$S_{\alpha,\alpha}(t) = \frac{1}{\pi}\int_0^\infty e^{-rt}K_{\alpha,\alpha}(r)dr - \frac{2}{\alpha}\rho^{(1-\alpha)/\alpha}e^{\rho^{1/\alpha}t\cos(\pi/\alpha)}\cos\left(\rho^{1/\alpha}t\sin\left(\frac{\pi}{\alpha}\right)+\frac{\pi}{\alpha}\right), \ \text{if } 1 \leq \alpha < 2. \tag{3.14}$$

Remark 3.12. *By* [4, pp. 3700]*, if $1 < \alpha = \beta < 2$, then*

$$\int_0^\infty |e_{\alpha,\alpha}(t)|dt \leq \frac{2}{\alpha\rho} - \frac{1}{\rho} - \frac{2}{\alpha\rho}\frac{1}{\cos(\pi/\alpha)} := l(\alpha,\rho). \tag{3.15}$$

4. Bounded mild solutions to semilinear case

In this section, we consider the semilinear differential equation

$$D^\alpha u(t) = Au(t) + (a \ast Au)(t) + f(t,u(t)), \quad t \in \mathbb{R}, \tag{4.16}$$

where A is the generator of an α-resolvent family. Motivated by the section 3, we define the concept of mild solution to equation (4.16) as follows.

Definition 4.13. *Let $\alpha > 0$ and A be the generator of an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$. A function $u \in C(\mathbb{R},X)$ is called a* mild solution *to equation (4.16) if the function $s \mapsto S_\alpha(t-s)f(s,u(s))$ is integrable on $(-\infty,t)$ for each $t \in \mathbb{R}$ and*

$$u(t) = \int_{-\infty}^t S_\alpha(t-s)f(s,u(s))ds, \quad t \in \mathbb{R}. \tag{4.17}$$

The following is the main result in this section.

Theorem 4.14. *Assume that A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ such that*

$$||S_\alpha(t)|| \leq \phi_\alpha(t), \quad \text{for all } t \geq 0 \quad \text{with } \phi_\alpha \in L^1(\mathbb{R}_+).$$

If $f \in \mathcal{M}(\mathbb{R} \times X, X)$ satisfies

$$||f(t,u) - f(t,v)|| \leq L||u - v||, \text{ for all } t \in \mathbb{R}, \text{ and } u, v \in X, \tag{4.18}$$

where $L < ||\phi_\alpha||_1^{-1}$. Then the equation (4.16) has a unique mild solution $u \in \mathcal{M}(X)$.

Proof. Define the operator $F : \mathcal{M}(X) \to \mathcal{M}(X)$ by

$$(F\phi)(t) := \int_{-\infty}^{t} S_\alpha(t-s) f(s, \phi(s))\, ds, \quad t \in \mathbb{R}. \tag{4.19}$$

By Theorems 2.1 and 2.2, F is well defined. For $\phi_1, \phi_2 \in \mathcal{M}(X)$ and $t \in \mathbb{R}$ we have:

$$\begin{aligned} ||(F\phi_1)(t) - (F\phi_2)(t)|| &\leq \int_{-\infty}^{t} ||S_\alpha(t-s)[f(s,\phi_1(s)) - f(s,\phi_2(s))]||ds \\ &\leq \int_{-\infty}^{t} L\|S_\alpha(t-s)\| \cdot \|\phi_1(s) - \phi_2(s)\|ds \\ &\leq L\|\phi_1 - \phi_2\|_\infty \int_{-\infty}^{t} \phi_\alpha(t-s)ds \\ &= L\|\phi_1 - \phi_2\|_\infty ||\phi_\alpha||_1. \end{aligned}$$

This prove that F is a contraction, so by the Banach fixed point theorem there exists a unique $u \in \mathcal{M}(X)$ such that $Fu = u$. □

The following result is a consequence of Theorem 4.14, Lemmas 3.9 and 3.10, and it is an extension of the case $\alpha = 1$ proved in [10, Theorem 3.2].

Corollary 4.15. *Let $\rho > 0$ be a real number. Let $f \in \mathcal{M}(\mathbb{R} \times X, X)$ such that*

$$||f(t,u) - f(t,v)|| \leq L||u - v||, \text{ for all } t \in \mathbb{R}, \text{ and } u, v \in X, \tag{4.20}$$

where $L < l(\alpha,\rho)^{-1}$, and $l(\alpha,\rho)$ is defined by (3.15). Then, for all $0 < \alpha < 2$ the equation

$$D^\alpha u(t) = -\rho u(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

has a unique mild solution $u \in \mathcal{M}(X)$ which is given by

$$u(t) = \int_{-\infty}^{t} S_{\alpha,\alpha}(t-s) f(s, u(s))ds,$$

where $S_{\alpha,\alpha}$ is defined by (3.13) if $0 < \alpha < 1$ and by (3.14) if $1 \leq \alpha < 2$.

A different condition is considered in the following result. Recall that an strongly continuous family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be uniformly bounded if there exists a constant $M > 0$ such that $||S(t)|| \leq M$ for all $t \geq 0$.

Theorem 4.16. *Assume that A generates an α-resolvent uniformly bounded family $\{S_\alpha(t)\}_{t\geq 0}$ on a Banach space X. If $f \in \mathcal{M}(\mathbb{R} \times X, X)$ satisfies*

$$||f(t,u) - f(t,v)|| \leq L(t)||u - v||, \text{ for all } t \in \mathbb{R}, \text{ and } u, v \in X, \tag{4.21}$$

where $L \in L^1(\mathbb{R}_+) \cap BC(\mathbb{R}_+)$. Then the equation (4.16) has a unique mild solution $u \in \mathcal{M}(X)$.

Proof. Define the operator F as in (4.19). For $\phi_1, \phi_2 \in \mathcal{M}(X)$ and $t \in \mathbb{R}$ we have:

$$\begin{aligned} ||(F\phi_1)(t) - (F\phi_2)(t)|| &\le \int_{-\infty}^{t} ||S(t-s)[f(s,\phi_1(s)) - f(s,\phi_2(s))]||ds \\ &\le M||\phi_1 - \phi_2||_\infty \int_0^\infty L_f(t-\tau)d\tau \\ &= M||\phi_1 - \phi_2||_\infty \int_{-\infty}^{t} L_f(s)ds. \end{aligned}$$

In general we get

$$\begin{aligned} ||(F^n\phi_1)(t) - (F^n\phi_2)(t)|| &\le ||\phi_1 - \phi_2||_\infty \frac{M^n}{(n-1)!}\left(\int_{-\infty}^{t} L_f(s)\left(\int_{-\infty}^{s} L_f(\tau)d\tau\right)^{n-1} ds\right) \\ &\le ||\phi_1 - \phi_2||_\infty \frac{M^n}{n!}\left(\int_{-\infty}^{t} L_f(s)ds\right)^n \\ &\le ||\phi_1 - \phi_2||_\infty \frac{(||L_f||_1 M)^n}{n!}. \end{aligned}$$

Hence, since $\frac{(||L_f||_1 M)^n}{n!} < 1$ for n sufficiently large, by the contraction principle F has a unique fixed point $u \in \mathcal{M}(X)$.

□

Example 4.17.

Let $A = -\rho I$ and $a(t) = \frac{\rho}{4}\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, where $0 < \alpha < 1$ and $\rho > 0$. From equation (4.16) we have

$$D^\alpha u(t) = -\rho u(t) - \rho \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds + f(t,u(t)), \quad t \in \mathbb{R}. \tag{4.22}$$

Using the Laplace transform, we obtain from Remark 2.4 that A generates an $\alpha-$resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ such that

$$\hat{S}_\alpha(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + \rho/2)^2} = \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + \rho/2)} \cdot \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + \rho/2)}.$$

Hence, $S_\alpha(t) = (r * r)(t)$, where $r(t) = t^{\frac{\alpha}{2}-1}E_{\alpha,\alpha/2}(-\frac{\rho}{2}t^\alpha)$. By Lemma 3.9, $S_\alpha \in L^1(\mathbb{R}_+)$. Therefore, if $f \in \mathcal{M}(\mathbb{R}\times X, X)$ and

$$||f(t,u) - f(t,v)|| \le L\,||u - v||, \text{ for all } t \in \mathbb{R}, \text{ and } u, v \in X,$$

where $L < ||S_\alpha||^{-1}$, then, there exists a unique mild solution $u \in \mathcal{M}(X)$ of equation (4.22), by Theorem 4.14.

Example 4.18.

Take $\alpha = 1/2$, $A = -\rho I$ and $a(t) = \gamma e^{-\beta t}$, with $\beta, \rho > 0$ and $\gamma \in \mathbb{R}$ in equation (4.16), that is

$$D^{\frac{1}{2}} u(t) = -\rho u(t) - \gamma\rho \int_{-\infty}^{t} e^{-\beta(t-s)} u(s)ds + f(t,u(t)), \quad t \in \mathbb{R}. \tag{4.23}$$

Using the Laplace transform and Remark 2.4, we have that A generates an $1/2-$resolvent family $\{S_{1/2}(t)\}_{t\geq 0}$ such that

$$\begin{aligned}\hat{S}_{1/2}(\lambda) &= \frac{\lambda+\beta}{\lambda^{3/2}+\lambda\rho+\lambda^{1/2}\beta+\rho(\beta+\gamma)}\\ &= \frac{\lambda+\beta}{(\lambda^{1/2}-r_1)(\lambda^{1/2}-r_2)(\lambda^{1/2}-r_3)},\end{aligned}$$

where r_1, r_2, r_3 are the roots (real or complex) of

$$z^3+\rho z^2+\beta z+\rho(\beta+\gamma). \tag{4.24}$$

Observe that

$$\hat{S}_{1/2}(\lambda) = \frac{\lambda^{1/2-1/6}}{(\lambda^{1/2}-r_1)}\cdot\frac{\lambda^{1/2-1/6}}{(\lambda^{1/2}-r_2)}\cdot\frac{\lambda^{1/2-1/6}}{(\lambda^{1/2}-r_3)}+\frac{\beta}{(\lambda^{1/2}-r_1)(\lambda^{1/2}-r_2)(\lambda^{1/2}-r_3)},$$

and therefore,

$$S_{1/2}(t) = (R_1*R_2*R_3)(t)+\beta(T_1*T_2*T_3)(t),$$

where $R_i(t) = t^{1/6-1}E_{1/2,1/6}(r_i t^{1/2})$ and $T_i(t) = t^{1/2-1}E_{1/2,1/2}(r_i t^{1/2})$, $i = 1,2,3$. If $r_i < 0$, $(i = 1,2,3)$ then $S_{1/2} \in L^1(\mathbb{R}_+)$ by Lemma 3.9. Now, we want to find conditions on ρ, β and γ to ensure that the roots of Eq. (4.24) are negative.

We recall that the roots r_1, r_2, r_3 of Eq. (4.24) verify

$$r_1+r_2+r_3 = -\rho, \tag{4.25}$$

$$r_1r_2+r_2r_3+r_1r_3 = \beta, \tag{4.26}$$

$$r_1r_2r_3 = -\rho(\beta+\gamma), \tag{4.27}$$

that the discriminant of Eq. (4.24) is given by

$$D := \rho^2(\beta+\gamma)[18\beta-4\rho^2-27(\beta+\gamma)]+\beta^2(\rho^2-4\beta), \tag{4.28}$$

and if $D \geq 0$, then the equation (4.24) has three real roots.

Assume that $D \geq 0$. Observe that if $\gamma > -\beta$, then $r_1r_2r_3 = -\rho(\beta+\gamma) < 0$. In this case, either all roots of Eq. (4.24) are negative, or one root of Eq. (4.24) is negative (say r_1) and the others, r_2, r_3, are positive. Using (4.25)-(4.27) we will see that this last case gives us a contradiction. Multiplying (4.26) by r_1, r_2 and r_3, we have

$$\begin{aligned}r_1^2r_2+r_1r_2r_3+r_1^2r_3 &= \beta r_1,\\ r_1r_2^2+r_2^2r_3+r_1r_2r_3 &= \beta r_2,\\ r_1r_2r_3+r_2r_3^2+r_1r_3^2 &= \beta r_3.\end{aligned}$$

Thus, by (4.25) and (4.27) we obtain,

$$r_1^2(r_2+r_3)-r_2^2(r_1+r_3)-r_3^2(r_1+r_2)-r_1r_2r_3 = \beta(-\rho-2(r_2+r_3)) < 0.$$

Since $r_1 < 0$, $r_2, r_3 > 0$, and $r_1+r_2+r_3 = -\rho$ then $r_1+r_3 < 0$ and $r_1+r_2 < 0$. This is gives us a contradiction. Therefore, if

$$D \geq 0 \quad \text{and} \quad \gamma > -\beta, \tag{4.29}$$

then all roots r_1, r_2, r_3 of (4.24) are negative.

We conclude that if ρ, β and γ verify the condition (4.29) and $f \in \mathcal{M}(\mathbb{R} \times X, X)$ satisfy

$$||f(t,u) - f(t,v)|| \leq L\,||u - v||, \text{ for all } t \in \mathbb{R}, \text{ and } u, v \in X,$$

where $L < ||S_\alpha||^{-1}$, then by Theorem 4.14, there exists a unique mild solution $u \in \mathcal{M}(X)$ of equation (4.23).

A description of the area in the plane where we can choose β and γ in order to have uniform integrability of $S_{1/2}(t)$ for $\rho > 0$, is shown in the hatched sector of the following figure. In particular, we note that if $\rho = 1$, then the point $(\gamma, \beta) = (-0.24, 0.23)$ belongs to the hatched area. In this case $r_1 = -0.05258063414$, $r_2 = -0.2887319147$ and $r_3 = -0.6586874512$. Similarly, if $\rho = 2$, $\gamma = -0.8$ and $\beta = 0.81$, then $r_1 = -0.02638799777$, $r_2 = -0.5221924489$, and $r_3 = -1.451419553$.

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