ASYMPTOTIC BEHAVIOR AND REPRESENTATION OF SOLUTIONS TO A VOLTERRA KIND OF EQUATION WITH A SINGULAR KERNEL

RODRIGO PONCE AND MAHAMADI WARMA

ABSTRACT. Let A a densely defined closed, linear ω-sectorial operator of angle $\theta \in [0, \frac{\pi}{2})$ on a Banach space X, for some $\omega \in \mathbb{R}$. We give an explicit representation (in terms of some special functions) and study the precise asymptotic behavior as time goes to infinity of solutions to the following diffusion equation with memory: $u'(t) = Au(t) + (\kappa * Au)(t)$, $t > 0$, $u(0) = u_0$, associated with the (possible) singular kernel $\kappa(t) = \alpha e^{-\beta t} \frac{t^{\mu-1}}{\Gamma(\mu)}$, $t > 0$, where $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$ and $0 < \mu < 1$.

1. INTRODUCTION

In the present paper, we consider the following Volterra kind of system

$$\begin{cases} u'(t) = Au(t) + \int_0^t \kappa(t-s)Au(s)ds, \ t > 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where A with domain $D(A)$ is a densely defined linear sectorial operator on a Banach space X, u_0 is a given function in X and the possible (singular) kernel κ is given by

$$\kappa(t) = \alpha e^{-\beta t} \frac{t^{\mu-1}}{\Gamma(\mu)}, \quad t > 0, \tag{1.2}$$

with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$ and $0 < \mu < 1$. We mention that the kernel $\kappa \in L^1_{\text{loc}}([0, \infty))$, and also belongs to $L^1((0, \infty))$ if $\beta > 0$, but it is not in $W^{1,1}_{\text{loc}}([0, \infty))$ if $0 < \mu < 1$. It is straightforward to verify that the system (1.1) is equivalent to the integral equation:

$$u(t) = u_0 + \int_0^t (1 + 1 * \kappa)(t-s)Au(s)ds, \quad t \geq 0. \tag{1.3}$$

By [20, Chapter I], the well-posedness of the system (1.1) (or equivalently, the integral equation (1.3)) is equivalent to the existence of a family of bounded linear operators $(S^\mu_{\alpha,\beta}(t))_{t \geq 0}$ on X, which we shall call (α, β, μ)-**resolvent family**, verifying the following properties:

- $S^\mu_{\alpha,\beta}(0) = I$ and $S^\mu_{\alpha,\beta}(\cdot)$ is strongly continuous on $[0, \infty)$.
- $S^\mu_{\alpha,\beta}(t)x \in D(A)$ and $S^\mu_{\alpha,\beta}(t)Ax = AS^\mu_{\alpha,\beta}(t)x$ for all $x \in D(A)$ and $t \geq 0$.
- For all $x \in D(A)$ and $t \geq 0$, the resolvent equation holds:

$$S^\mu_{\alpha,\beta}(t)x = x + \int_0^t (1 + 1 * k)(t-s)AS^\mu_{\alpha,\beta}(s)x \ ds. \tag{1.4}$$

In that case, for every $u_0 \in X$, the unique (mild) solution of (1.1) is given by

$$u(t) = S^\mu_{\alpha,\beta}(t)u_0, \quad t \geq 0. \tag{1.5}$$

We give the idea how to get (1.5) without further details. In fact, uniqueness of solutions is easy to establish. If $u(t)$ and $S_{\alpha,\beta}^{\mu}(t)$ are exponentially bounded, then taking the Laplace transform of both sides of the first equation in (1.1), we get that

$$\lambda\widehat{u}(\lambda) - u(0) = A\widehat{u}(\lambda) + \frac{\alpha}{(\lambda+\beta)^{\mu}}A\widehat{u}(\lambda).$$

Thus

$$\frac{(\lambda+\beta)^{\mu}+\alpha}{(\lambda+\beta)^{\mu}}\left(\frac{\lambda(\lambda+\beta)^{\mu}}{(\lambda+\beta)^{\mu}+\alpha} - A\right)\widehat{u}(\lambda) = [g(\lambda)]^{-1}\Big(h(\lambda) - A\Big)\widehat{u}(\lambda) = u_0, \qquad (1.6)$$

where we have set

$$g(\lambda) := \frac{(\lambda+\beta)^{\mu}}{(\lambda+\beta)^{\mu}+\alpha} \quad \text{and} \quad h(\lambda) := \frac{\lambda(\lambda+\beta)^{\mu}}{(\lambda+\beta)^{\mu}+\alpha} = \lambda g(\lambda).$$

Note that $g(\lambda)$ and $h(\lambda)$ are well-defined for all λ with $\mathrm{Re}(\lambda) > 0$. Taking also the Laplace transform of both sides of (1.4) with $x = u_0$ we get that

$$\widehat{S}_{\alpha,\beta}^{\mu}(\lambda)u_0 = \frac{1}{\lambda}u_0 + \left(\frac{1}{\lambda} + \frac{\widehat{k}(\lambda)}{\lambda}\right)A\widehat{S}_{\alpha,\beta}^{\mu}(\lambda)u_0.$$

Calculating, we obtain that

$$\frac{(\lambda+\beta)^{\mu}+\alpha}{(\lambda+\beta)^{\mu}}\left(\frac{\lambda(\lambda+\beta)^{\mu}}{(\lambda+\beta)^{\mu}+\alpha} - A\right)\widehat{S}_{\alpha,\beta}^{\mu}(\lambda)u_0 = [g(\lambda)]^{-1}\Big(h(\lambda) - A\Big)\widehat{S}_{\alpha,\beta}^{\mu}(\lambda)u_0 = u_0. \qquad (1.7)$$

It follows from (1.6) and (1.7) that if $h(\lambda) \in \rho(A)$, then

$$\widehat{u}(\lambda) = \widehat{S}_{\alpha,\beta}^{\mu}(\lambda)u_0 = g(\lambda)\Big(h(\lambda) - A\Big)^{-1}u_0. \qquad (1.8)$$

Now taking the inverse Laplace transform of (1.8) we get that the unique (mild) solution of (1.1) is given by (1.5). For more details on this topic we refer to the monograph [20].

Next, we assume that the system (1.1) is well-posed and we denote by $S_{\alpha,\beta}^{\mu}$ the associated (α,β,μ)-resolvent family. Throughout the remainder of the paper, without any mention, the norms are taken in the Banach space X.

Definition 1.1. *We shall say that $S_{\alpha,\beta}^{\mu}$ is* **exponentially bounded, or of type** $(M,\widetilde{\omega})$*, if there exist two constants $M > 0$ and $\widetilde{\omega} \in \mathbb{R}$ such that*

$$\|S_{\alpha,\beta}^{\mu}(t)\| \leq Me^{\widetilde{\omega}t}, \quad \forall\, t \geq 0. \qquad (1.9)$$

The resolvent family $S_{\alpha,\beta}^{\mu}$ will be said to be **uniformly exponentially stable** *if (1.9) holds with some constants $M > 0$ and $\widetilde{\omega} < 0$.*

As we have mentioned above, the well-posedness of the system (1.1) and many other properties of resolvent families have been intensively studied in the monograph [20]. Similar problems have been also considered in [1, 3, 4, 7, 12, 14, 15, 18, 19] and their references. The asymptotic behavior as time goes to ∞ of solutions to some similar integro-differential equations in finite dimension involving kernels as the one in (1.2) have been also investigated in [21].

Fractional order operators have recently emerged as a modeling alternative in various branches of science and technology. In fact, in many situations, the fractional models reflect better the behavior of the system both in the deterministic and stochastic contexts. They usually describe anomalous phenomena. A number of stochastic models for explaining anomalous diffusion have been introduced in the literature; among them we mention the fractional Brownian motion; the continuous time random walk; the Lévy flights; the Schneider grey Brownian motion; and more generally, random walk models based on evolution equations of single and distributed fractional order in space. In general, a fractional diffusion operator corresponds to a diverging jump length

variance in the random walk. We refer to [3, 7, 8, 9, 11] and the references therein for the derivations and applications of fractional partial differential equations.

The main concerns in the present paper are the following:

(1) Study the precise asymptotic behavior as time goes to infinity of solutions to the system (1.1). That is, we would like to characterize

$$\inf \left\{ \widetilde{\omega} \in \mathbb{R} \ \text{ such that the estimate (1.9) is satisfied} \right\}.$$

Of particular interest will be to find some precise conditions on the parameters α, β, μ and the operator A that shall imply the uniform exponential stability of the family $S_{\alpha,\beta}^{\mu}$, and hence, of solutions to the system (1.1).

(2) Find an explicit representation of solutions to the system (1.1) in terms of some special functions. The Mittag-Leffler functions and its generalizations will be the natural candidates.

We mention that if $\mu = 1$ and $\beta > 0$ in (1.2), that is, $\kappa(t) = \alpha e^{-\beta t}$, then the asymptotic behavior of the associated resolvent family has been completely studied in [2] by using some semigroups method. See also [6, Chapter VI, Section 7] where the well-posedness of the system (1.1) for a general kernel $\kappa \in W^{1,1}((0, \infty))$ has been obtained by using again the method of semigroups.

One of our concerns is to extend the results contained in [2] to the case $0 < \mu < 1$, where the method of semigroups cannot be used. Note that in our situation, $\kappa \notin W_{\text{loc}}^{1,1}([0, \infty))$. If $\beta = 0$ and $0 < \mu < 1$, then it is well-known (see e.g. [5, 17, 11, 13, 18] and their references) that the corresponding $(\alpha, 0, \mu)$-resolvent family is never uniformly exponentially stable. In this case, solutions may decay but only at most polynomial. We shall see here that the situation is different if $\beta > 0$. More precisely, assuming that the operator A is ω-sectorial of angle θ (see Section 2 below for the definition) for some $\omega < 0$, $0 \leq \theta \leq \widetilde{\mu} \frac{\pi}{2}$, where $0 < \mu < \widetilde{\mu} < 1$, and that $\beta + \omega \leq 0$, we obtain the following result:

(a) If $\alpha > 0$, then the family is uniformly exponentially stable with $\widetilde{\omega} = -\beta$.

(b) If $\alpha < 0$ and $\alpha + \beta^{\mu} \geq |\alpha|$, then the family is exponentially bounded with exponential bound $\widetilde{\omega} = -\left(\beta - (\alpha\omega)^{\frac{1}{\mu+1}} \right)$. This will imply the uniform exponential stability of the family if in addition $\beta^{\mu+1} > \alpha\omega$.

These results are contained in Theorems 2.3 and 2.4 below.

The rest of the paper is structured as follows. In Section 2 we state the main results where some generation results of (α, β, μ)-resolvent families and the precise asymptotic behavior as time goes to infinity of the resolvent family have been obtained. Section 3 contains the proof of the generation theorem. In Section 4 we give the proof of the asymptotic behavior as time goes to infinity of the resolvent family. Finally in Section 5, assuming that $A = \rho I$ for some $\rho \in \mathbb{R}$, or A is a self-adjoint operator on $L^2(\Omega)$ (where $\Omega \subset \mathbb{R}^N$ is a bounded open set) with a compact resolvent, we obtain an explicit representation of solutions to the system (1.1) (and hence, of the associated resolvent family) in terms of generalized Mittag-Leffler functions.

2. Main results

Let X be a Banach space with norm $\| \cdot \|$. For a closed, linear and densely defined operator A on X, we denote by $\sigma(A)$ and $\rho(A)$ the spectrum, and the resolvent set of A, respectively.

The operator A is said to be ω-**sectorial of angle** θ, if there exist $\theta \in [0, \pi/2)$ and $\omega \in \mathbb{R}$ such that

$$\omega + \Sigma_\theta := \left\{ \omega + \lambda, \ \lambda \in \mathbb{C} : |\arg(\lambda)| < \theta + \frac{\pi}{2} \right\} \subset \rho(A),$$

and there is a constant $M > 0$ such that one has the estimate

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \quad \forall \, \lambda \in \{\omega + \Sigma_\theta\} \setminus \{\omega\}.$$

If $\omega = 0$, then we shall only say that A is **sectorial of angle** θ. More details on sectorial operators can be found in [6, 10] and the references therein.

In this section we state the main results of the paper. We start with the generation theorem.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$ and $0 < \mu < 1$. Assume that any one of the following two conditions holds.*

(a) *$\alpha > 0$ and A is a sectorial operator of angle $\mu\frac{\pi}{2}$.*
(b) *$\alpha < 0$, $\alpha + \beta^\mu \geq |\alpha|$ and A is a sectorial operator of angle $\mu\frac{\pi}{2}$.*

Then, A generates an (α, β, μ)-resolvent family $(S_{\alpha,\beta}^\mu(t))_{t \geq 0}$ of type $(M, \widetilde{\omega})$ for every $\widetilde{\omega} > 0$.

Corollary 2.2. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$, $0 < \mu < 1$ and $\omega \in \mathbb{R}$. Assume that any one of the following two conditions holds.*

(a) *$\alpha > 0$ and A is an ω-sectorial operator of angle $\mu\frac{\pi}{2}$.*
(b) *$\alpha < 0$, $\alpha + \beta^\mu \geq |\alpha|$ and A is an ω-sectorial operator of angle $\mu\frac{\pi}{2}$.*

Then, A generates an (α, β, μ)-resolvent family $(S_{\alpha,\beta}^\mu(t))_{t \geq 0}$ of type $(M, \widetilde{\omega})$ for some $\widetilde{\omega} > 0$.

As we have mentioned above, the asymptotic behavior, as time goes to infinity, of the resolvent family for the case $\mu = 1$ has been completely studied in [2]. Therefore, we shall concentrate on the case $0 < \mu < 1$.

The following estimates of the resolvent family is the second main result of the paper.

Theorem 2.3. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$, $0 < \mu < \widetilde{\mu} < 1$ and $\omega < 0$. Assume that $\beta + \omega \leq 0$. Then the following assertions hold.*

(a) *If $\alpha > 0$ and A is ω-sectorial of angle $\widetilde{\mu}\frac{\pi}{2}$, then there exists a constant $C > 0$ such that the resolvent family $(S_{\alpha,\beta}^\mu(t))_{t \geq 0}$ generated by A satisfies the estimate*

$$\|S_{\alpha,\beta}^\mu(t)\| \leq Ce^{-\beta t}, \quad \forall \, t \geq 0. \tag{2.1}$$

(b) *If $\alpha < 0$, $\alpha + \beta^\mu \geq |\alpha|$, and A is ω-sectorial of angle $\widetilde{\mu}\frac{\pi}{2}$, then there exists a constant $C > 0$ such that the resolvent family $(S_{\alpha,\beta}^\mu(t))_{t \geq 0}$ generated by A satisfies the estimate*

$$\|S_{\alpha,\beta}^\mu(t)\| \leq C\left(1 + \alpha\omega t^{\mu+1}\right) e^{-(\beta - (\alpha\omega)^{\frac{1}{\mu+1}})t}, \quad \forall \, t \geq 0. \tag{2.2}$$

We conclude this section by considering also the case $\beta + \omega > 0$.

Theorem 2.4. *Let $\alpha > 0$, $\beta > 0$, $0 < \mu < 1$ and $\omega < 0$. Assume that $\beta + \omega > 0$. If A is an ω-sectorial operator of angle $\widetilde{\mu}\frac{\pi}{2}$, then there exists a constant $C > 0$ such that the resolvent family $(S_{\alpha,\beta}^\mu(t))_{t \geq 0}$ generated by A satisfies the estimate*

$$\|S_{\alpha,\beta}^\mu(t)\| \leq Ce^{\omega t}, \quad t \geq 0. \tag{2.3}$$

3. Proof of Theorem 2.1 and Corollary 2.2

Before given the proof of the generation theorem, we need some intermediate results. First, we recall the following fundamental result taken from [20, Chapter I, Theorem 1.3].

Theorem 3.1. *Let $\widetilde{\omega} \in \mathbb{R}$ and $\mathfrak{a} \in L_{\mathrm{loc}}^1([0, \infty))$ satisfy*

$$\int_0^\infty e^{-\widetilde{\omega}t}|\mathfrak{a}(t)| \, dt < \infty. \tag{3.1}$$

Let A with domain $D(A)$ be a linear operator on X with dense domain. Then A generates a resolvent family (associated with $\mathfrak{a}$) *of type $(M, \widetilde{\omega})$ on X if and only if the following two conditions hold.*

(a) $\widehat{\mathfrak{a}}(\lambda) \neq 0$ *and* $\frac{1}{\widehat{\mathfrak{a}}(\lambda)} \in \rho(A)$ *for all* $\lambda > \widetilde{\omega}$.

(b) *The mapping* $\lambda \mapsto H(\lambda) := \frac{1}{\lambda}\left(I - \widehat{\mathfrak{a}}(\lambda)A\right)^{-1}$ *satisfies the estimate*

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\lambda - \widetilde{\omega})^{n+1}}, \quad \lambda > \widetilde{\omega}, \ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $H^{(n)}(\lambda) = \frac{d^n H}{d\lambda^n}(\lambda)$, *and* $M > 0$ *is a constant.*

The following result will be also useful.

Lemma 3.2. *Let $\alpha \in \mathbb{R}$, $\beta \geq 0$, $0 < \mu < 1$ and define $g(\lambda) := \frac{(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu + \alpha}$ for all λ with $\mathrm{Re}(\lambda) > 0$. Then the following assertions hold.*

(a) *If $\alpha \geq 0$, then $|g(\lambda)| \leq 1$.*

(b) *If $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$, then $|g(\lambda)| \leq \frac{\beta^\mu}{\alpha + \beta^\mu} \leq \frac{\beta^\mu}{|\alpha|}$.*

Proof. Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $0 < \mu < 1$.

(a) This assertion is obvious.

(b) Now, assume that $\alpha < 0$ and $\beta^\mu + \alpha \geq |\alpha|$. Observe that

$$|g(\lambda)| = \left|1 - \frac{\alpha}{(\lambda+\beta)^\mu + \alpha}\right| \leq 1 + \frac{|\alpha|}{|(\lambda+\beta)^\mu - \beta^\mu + (\alpha + \beta^\mu)|}. \tag{3.2}$$

We claim that

$$\mathrm{Re}[(\lambda+\beta)^\mu - \beta^\mu] > 0 \ \text{ for all } \ \mathrm{Re}(\lambda) > 0. \tag{3.3}$$

We show first that

$$\mathrm{Re}[(\lambda+\beta)^\mu] > (\mathrm{Re}(\lambda) + \beta)^\mu \ \text{ for all } \ \mathrm{Re}(\lambda) > 0. \tag{3.4}$$

Indeed, let $z := \lambda + \beta$. We have to show that $\mathrm{Re}(z^\mu) > (\mathrm{Re}(z))^\mu$, that is, $|z|^\mu \cos(\theta\mu) > (|z|\cos(\theta))^\mu$, where $\mathrm{Re}(z) = |z|\cos(\theta)$ with $|\theta| < \frac{\pi}{2}$. Let $f(\mu) := \cos(\mu\theta) - \cos^\mu(\theta)$, $0 \leq \mu \leq 1$. Then

$$f''(\mu) = -\theta^2 \cos(\mu\theta) - (\ln(\cos(\theta)))^2 \cos^\mu(\theta) < 0 \ \text{ for all } \ 0 < \mu < 1,$$

and this implies that f is a concave function. Since $f(0) = f(1) = 0$, we have that the graph of f is above the straight line joining the points $(0, f(0)) = (0, 0)$ and $(1, f(1)) = (1, 0)$. This implies that $f(\mu) \geq 0$ for all $0 \leq \mu < 1$. Thus, $\cos(\mu\theta) \geq \cos^\mu(\theta)$ for all $0 \leq \mu < 1$, and we have shown (3.4). The claim (3.3) follows from (3.4).

On the other hand, since by assumption $\alpha + \beta^\mu \geq |\alpha| > 0$, it follows from (3.3) that

$$|\mathrm{Re}[(\lambda+\beta)^\mu - \beta^\mu + (\alpha + \beta^\mu)]| > \alpha + \beta^\mu. \tag{3.5}$$

Using (3.5), we get from (3.2) that

$$|g(\lambda)| \leq 1 + \frac{|\alpha|}{\alpha + \beta^\mu} = \frac{\beta^\mu}{\alpha + \beta^\mu} \leq \frac{\beta^\mu}{|\alpha|},$$

and the proof is finished. $\qquad\square$

We also need the following result.

Lemma 3.3. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $\beta \geq 0$, $0 < \mu < 1$ and define the function*

$$h(\lambda) := \frac{\lambda(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu + \alpha}, \quad \lambda \in \mathbb{C}, \ \mathrm{Re}(\lambda) > 0. \tag{3.6}$$

Assume that any one of the following two conditions holds.

(a) $\alpha > 0$.

(b) $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$.

Then

$$| \arg(h(\lambda))| \leq (1 + \mu)| \arg(\lambda)|. \tag{3.7}$$

Proof. Let h be given by (3.6) where $\lambda = re^{i\theta}$ with $|\theta| < \frac{\pi}{2}$ and $r > 0$. Without any restriction we may assume that $\theta \geq 0$. Then

$$\arg(h(re^{i\theta})) = \mathrm{Im}(\ln(h(re^{i\theta}))) = \mathrm{Im} \int_0^\theta \frac{d}{dt}\ln(h(re^{it}))dt = \mathrm{Im} \int_0^\theta \frac{h'(re^{it})ire^{it}}{h(re^{it})}dt. \tag{3.8}$$

Moreover a simple calculation gives

$$\lambda\frac{h'(\lambda)}{h(\lambda)} = 1 + \left(\frac{\alpha\mu}{(\lambda+\beta)^\mu + \alpha} \right) \left(\frac{\lambda}{\lambda+\beta} \right).$$

(a) Assume that $\alpha > 0$. Then

$$\left| \lambda\frac{h'(\lambda)}{h(\lambda)} \right| \leq 1 + \mu \left| \frac{\alpha}{(\lambda+\beta)^\mu + \alpha} \right| \left| \frac{\lambda}{\lambda+\beta} \right| \leq 1 + \mu. \tag{3.9}$$

Using (3.9) we get from (3.8) that

$$\left| \mathrm{Im} \int_0^\theta \frac{h'(re^{it})ire^{it}}{h(re^{it})}dt \right| \leq \int_0^\theta \left| \frac{h'(re^{it})ire^{it}}{h(re^{it})} \right| dt \leq \int_0^\theta (1 + \mu)dt \leq (1 + \mu)\theta,$$

and we have shown (3.7).

(b) Now assume that $\alpha < 0$ and $\beta^\mu + \alpha \geq |\alpha|$. Then using (3.5) we get that

$$\left| \lambda\frac{h'(\lambda)}{h(\lambda)} \right| \leq 1 + \mu \left| \frac{\alpha}{(\lambda+\beta)^\mu + \alpha} \right| \left| \frac{\lambda}{\lambda+\beta} \right| \leq 1 + \frac{\mu|\alpha|}{\alpha+\beta^\mu} \leq 1 + \mu. \tag{3.10}$$

It follows from (3.8) and (3.10) that

$$\left| \mathrm{Im} \int_0^\theta \frac{h'(re^{it})ire^{it}}{h(re^{it})}dt \right| \leq \int_0^\theta \left| \frac{h'(re^{it})ire^{it}}{h(re^{it})} \right| dt \leq \int_0^\theta 1 + \frac{\mu|\alpha|}{\alpha+\beta^\mu}dt \leq (1 + \mu)\theta.$$

We have shown (3.7) and the proof is finished. $\qquad\qquad\square$

Proof of Theorem 2.1. First notice that if we compare with Theorem 3.1, then

$$\mathfrak{a}(t) = (1 + 1 * \kappa)(t), \quad t > 0,$$

so that its Laplace transform is given by

$$\widehat{\mathfrak{a}}(\lambda) = \frac{1}{\lambda} + \frac{\widehat{\kappa}(\lambda)}{\lambda} = \frac{(\lambda+\beta)^\mu + \alpha}{\lambda(\lambda+\beta)^\mu}.$$

It is clear that for every $\widetilde{\omega} > 0$ we have that

$$\int_0^\infty e^{-\widetilde{\omega}t}|\mathfrak{a}(t)| \, dt = \int_0^\infty e^{-\widetilde{\omega}t}|(1 + 1 * \kappa)(t)| \, dt < \infty.$$

Hence, we have to show that the two conditions in Theorem 3.1 are satisfied. It is easy to see that under the assumptions (a) or (b) we have that $\widehat{\mathfrak{a}}(\lambda) \neq 0$ for all λ with $\mathrm{Re}(\lambda) > 0$.

Next, we claim that

$$\frac{1}{\widehat{\mathfrak{a}}(\lambda)} = h(\lambda) = \frac{\lambda(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu + \alpha} \in \rho(A) \quad \text{for all} \ \lambda \ \text{with} \ \mathrm{Re}(\lambda) > 0. \tag{3.11}$$

It follows from Lemma 3.3 that in both cases (a) and (b), we have that

$$h(\lambda) \in \Sigma_{\frac{\pi\mu}{2}} \quad \text{for all} \ \lambda \ \text{with} \ \mathrm{Re}(\lambda) > 0.$$

This implies that the function $H(\lambda) = g(\lambda)(h(\lambda) - A)^{-1}$ is well defined, where $g(\lambda)$ is given by

$$g(\lambda) := \frac{(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} = \frac{h(\lambda)}{\lambda}.$$

Since A is a sectorial operator of angle $\mu\frac{\pi}{2}$ (in both cases (a) and (b)), we have that there exists a constant $M > 0$ such that for all λ with $\text{Re}(\lambda) > 0$,

$$\|\lambda H(\lambda)\| = |h(\lambda)| \left\|(h(\lambda) - A)^{-1}\right\| \leq M\frac{|h(\lambda)|}{|h(\lambda)|} = M. \tag{3.12}$$

Moreover a simple calculation gives that in both cases,

$$\lambda^2 H'(\lambda) = \mu\tfrac{\lambda}{\lambda+\beta}\lambda H(\lambda) - \mu\tfrac{\lambda}{\lambda+\beta}g(\lambda)\lambda H(\lambda) - \lambda^2 H(\lambda)^2$$
$$- \mu\tfrac{\lambda}{\lambda+\beta}\lambda^2 H(\lambda)^2 + g(\lambda)\lambda^2 H(\lambda)^2\mu\tfrac{\lambda}{\lambda+\beta}.$$

Note that in the case (b) we have that $\alpha + \beta^\mu \geq (1 + \mu)\alpha + \beta^\mu \geq |\alpha| + \mu\alpha > 0$. Since the function g is bounded (by Lemma 3.2), using (3.12), we get that there exists a constant $M_1 > 0$ such that

$$\|\lambda^2 H'(\lambda)\| \leq M_1 \quad \text{for all} \ \ \lambda \ \ \text{with} \ \ \text{Re}(\lambda) > 0. \tag{3.13}$$

Combining (3.12)-(3.13) we get that there exists a constant $M > 0$ such that

$$\|\lambda H(\lambda)\| + \|\lambda^2 H'(\lambda)\| \leq M \quad \text{for all} \ \ \lambda \ \ \text{with} \ \ \text{Re}(\lambda) > 0. \tag{3.14}$$

By [20, Proposition 0.1], the estimate (3.14) implies that

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{\lambda^{n+1}}, \quad \forall\, \lambda > 0, \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{3.15}$$

From (3.15) we also have that for every $\widetilde{\omega} > 0$,

$$\|H^{(n)}(\lambda)\| \leq \frac{Mn!}{(\lambda - \widetilde{\omega})^{n+1}}, \quad \forall\, \lambda > \widetilde{\omega}, \ n \in \mathbb{N}_0.$$

Finally, it follows from Theorem 3.1 that the operator A generates an (α, β, μ)-resolvent family $(S^\mu_{\alpha,\beta}(t))_{t\geq 0}$ of type $(M, \widetilde{\omega})$ and the proof is finished. $\qquad\square$

Proof of Corollary 2.2. Assume that $\alpha > 0$, or $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$, and that A is ω-sectorial of angle $\mu\frac{\pi}{2}$. The claim follows from the decomposition $A = (\omega I + A) - \omega I$, using Theorem 2.1 and the perturbation result of resolvent families contained in [16, Corollary 3.2]. The proof is finished. $\qquad\square$

We make some comments about the results obtained in Theorem 2.1 and Corollary 2.2.

Remark 3.4. We notice that even if we assume that the operator A is ω-sectorial for some $\omega < 0$, then uniform exponential stability of the family $(S^\mu_{\alpha,\beta}(t))_{t\geq 0}$ cannot be derived from Theorem 2.1 and Corollary 2.2. In fact, the integral condition (3.1) on the kernel $\mathfrak{a}(t) = (1 + 1 * \kappa)(t)$ is only satisfied for $\widetilde{\omega} > 0$. Therefore some new ideas are needed in the study of the uniform exponential stability of the family which is one of the main goals of the present paper.

4. Proof of Theorems 2.3 and 2.4

In this section we give the proof of Theorems 2.3 and 2.4.

Proof of Theorem 2.3. Let $\beta \geq 0$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu < \widetilde{\mu} < 1$ and $\omega < 0$.

(a) Since A is ω-sectorial of angle $0 < \theta \leq \widetilde{\mu}\frac{\pi}{2}$, it follows from Theorem 2.1 and Corollary 2.2 that A generates an (α, β, μ)-resolvent family $(S^\mu_{\alpha,\beta}(t))_{t\geq 0}$. Recall the function h given by

$$h(\lambda) := \frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha}.$$

We have that for every $\lambda \in \mathbb{C}$ such that $h(\lambda) \in \omega + \Sigma_\theta$, the resolvent $\left(\frac{\lambda(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu+\alpha}I - A\right)^{-1}$ exists and the Laplace transform of $S_{\alpha,\beta}^\mu(t)$ is given by

$$\widehat{S}_{\alpha,\beta}^\mu(\lambda) = \frac{(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu+\alpha}\left(\frac{\lambda(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu+\alpha}I - A\right)^{-1}. \tag{4.1}$$

By the uniqueness of the Laplace transform, it follows from (4.1) that

$$S_{\alpha,\beta}^\mu(t) = e^{-\beta t}G_{\alpha,\beta}^\mu(t) \text{ for all } t \geq 0, \tag{4.2}$$

where the Laplace transform of $G_{\alpha,\beta}^\mu(t)$ is given, for all $\lambda \in \mathbb{C}$ such that $h(\lambda-\beta) \in \omega + \Sigma_\theta$, by

$$\widehat{G}_{\alpha,\beta}^\mu(\lambda) = \frac{\lambda^\mu}{\lambda^\mu+\alpha}\left(\frac{(\lambda-\beta)\lambda^\mu}{\lambda^\mu+\alpha}I - A\right)^{-1}.$$

We notice that $h(\lambda) \in \omega + \Sigma_\theta$ if and only if $\widetilde{h}(\lambda) \in \omega + \Sigma_\theta$, where

$$h(\lambda-\beta) = \widetilde{h}(\lambda) := \frac{\lambda^\mu(\lambda-\beta)}{\lambda^\mu+\alpha}.$$

For all $\lambda \in \mathbb{C}$ with $\widetilde{h}(\lambda) \in \omega + \Sigma_\theta$, there exists a constant $M > 0$ such that we have the estimate

$$\left\|\left(\frac{(\lambda-\beta)\lambda^\mu}{\lambda^\mu+\alpha}I - A\right)^{-1}\right\| \leq \frac{M}{\left|\frac{(\lambda-\beta)\lambda^\mu}{\lambda^\mu+\alpha} - \omega\right|} = \frac{M|\lambda^\mu+\alpha|}{|\lambda^{\mu+1} - (\beta+\omega)\lambda^\mu - \alpha\omega|}. \tag{4.3}$$

Note that $\alpha\omega < 0$ and by hypothesis $-(\omega+\beta) \geq 0$. Let

$$g(t) := t^{\mu+1}, \ t > 0.$$

We exploit some ideas from the proof of [5, Theorem 1]. Recall that A is ω-sectorial of angle $\theta = \widetilde{\mu}\frac{\pi}{2}$. Using the inversion formula for the Laplace transform, we get that

$$G_{\alpha,\beta}^\mu(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\frac{\lambda^\mu}{\lambda^\mu+\alpha}\left(\frac{(\lambda-\beta)\lambda^\mu}{\lambda^\mu+\alpha}I - A\right)^{-1}d\lambda, \tag{4.4}$$

where γ is a positively oriented path whose support Γ is given by

$$\Gamma := \left\{\lambda \in \mathbb{C}: \ \lambda^{\mu+1} \text{ belongs to the boundary of } B_{\frac{1}{g(t)}}, \ t > 0\right\},$$

where for $\delta > 0$,

$$B_\delta := \{\delta + \Sigma_\theta\} \cup S_\phi,$$

and

$$S_\phi := \left\{\lambda \in \mathbb{C}: \ |\arg(\lambda)| < \phi\right\}, \ (\mu+1)\frac{\pi}{2} < \phi < \frac{\pi}{2} + \theta. \tag{4.5}$$

We claim that for all such λ given in (4.5), we have $h(\lambda) \in \omega + \Sigma_\theta$. Firstly, notice that if $\omega \leq 0$, hence $-\omega \geq 0$, then

$$|\arg(h(\lambda) - \omega)| \leq |\arg(h(\lambda))|. \tag{4.6}$$

Secondly, we take $\lambda = re^{i\phi_0}$ in the above sector. If $\text{Re}(\lambda) > 0$, then the hypotheses and Lemma 3.3 imply that (recall that $|\arg(\lambda)| \leq \frac{\pi}{2}$)

$$|\arg(h(\lambda))| \leq (1+\mu)|\arg(\lambda)| < (1+\mu)\frac{\pi}{2}.$$

Since λ is as in (4.5), it follows from the preceding inequality that

$$|\arg(h(\lambda))| \leq (1+\mu)|\arg(\lambda)| < (1+\mu)\frac{\pi}{2} < \phi < \frac{\pi}{2} + \theta.$$

Thus, as $\omega < 0$, using (4.6) we can deduce that $h(\lambda) \in \omega + \Sigma_\theta$.

Now, assume that $\mathrm{Re}(\lambda) < 0$ and λ belongs to this sector. First, we assume that $\alpha > 0$ and $0 < \mu < \frac{1}{2}$. Let $z := \lambda + \beta = Re^{i\varphi}$. Since $\beta > 0$, we have that $0 < \varphi = \arg(z) = \arg(\lambda + \beta) < \arg(\lambda) = \phi_0 < \pi$. A simply computation gives

$$\frac{(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha} = \frac{z^\mu}{z^\mu + \alpha} = \frac{R^{2\mu} + \alpha R^\mu \cos(\varphi\mu) + \alpha R^\mu \sin(\varphi\mu)i}{R^{2\mu} + 2\alpha R^\mu \cos(\varphi\mu) + \alpha^2}.$$

It follows from this identity that

$$\frac{\mathrm{Im}\left(\frac{z^\mu}{z^\mu + \alpha}\right)}{\mathrm{Re}\left(\frac{z^\mu}{z^\mu + \alpha}\right)} = \frac{\alpha \sin(\varphi\mu)}{R^\mu + \alpha \cos(\varphi\mu)} =: p(R).$$

Since $\alpha > 0$ and $0 < \mu < \frac{1}{2}$, it follows that the function $p(R)$ is differentiable for all $R \geq 0$ and

$$p'(R) = \frac{-\alpha\mu \sin(\varphi\mu) R^{\mu-1}}{(R^\mu + \alpha \cos(\varphi\mu))^2}.$$

From the assumptions on the parameters α and μ we obtain $p'(R) \leq 0$ for all $R \geq 0$, which implies that p is a decreasing function. Thus,

$$p(R) \leq p(0) = \tan(\varphi\mu) = \frac{\sin(\varphi\mu)}{\cos(\varphi\mu)} = \frac{\mathrm{Im}(e^{i\varphi\mu})}{\mathrm{Re}(e^{i\varphi\mu})}.$$

Since $\arctan(\cdot)$ is an increasing function, it follows that

$$\arg\left(\frac{z^\mu}{z^\mu + \alpha}\right) = \arctan\left(\frac{\mathrm{Im}\left(\frac{z^\mu}{z^\mu + \alpha}\right)}{\mathrm{Re}\left(\frac{z^\mu}{z^\mu + \alpha}\right)}\right) \leq \arctan\left(\frac{\mathrm{Im}(e^{i\varphi\mu})}{\mathrm{Re}(e^{i\varphi\mu})}\right) = \arg(e^{i\varphi\mu}) = \varphi\mu < \mu\arg(\lambda).$$

Hence,

$$\arg\left(\frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha}\right) = \arg(\lambda) + \arg\left(\frac{(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha}\right) < \arg(\lambda) + \mu\arg(\lambda) = (1 + \mu)\arg(\lambda).$$

On the other hand, if $\alpha > 0$ and $\frac{1}{2} \leq \mu < 1$, then we notice that we may assume that $0 < \varphi < \frac{\pi}{2}$. An easy computation gives

$$\frac{\mathrm{Im}(z^\mu + \alpha)}{\mathrm{Re}(z^\mu + \alpha)} = \frac{R^\mu \sin(\varphi\mu)}{R^\mu \cos(\varphi\mu) + \alpha} =: q(R)$$

and

$$q'(R) = \alpha\mu \frac{R^{\mu-1} \sin(\varphi\mu)}{(R^\mu \cos(\varphi\mu) + \alpha)^2} \geq 0,$$

for all $R \geq 0$, which means that q is an increasing function, and therefore $q(R) \geq q(0) = 0$. Hence,

$$\frac{\mathrm{Im}(z^\mu + \alpha)}{\mathrm{Re}(z^\mu + \alpha)} \geq 0.$$

Thus,

$$\arg((\lambda + \beta)^\mu + \alpha) = \arg(z^\mu + \alpha) = \arctan\left(\frac{\mathrm{Im}(z^\mu + \alpha)}{\mathrm{Re}(z^\mu + \alpha)}\right) \geq 0.$$

Since $\beta > 0$, we have that

$$\arg\left(\frac{\lambda(\lambda + \beta)^\mu}{(\lambda + \beta)^\mu + \alpha}\right) = \arg(\lambda) + \mu\arg(\lambda + \beta) - \arg\left((\lambda + \beta)^\mu + \alpha\right) \leq \arg(\lambda) + \mu\arg(\lambda) = (1 + \mu)\arg(\lambda).$$

In both cases, we get

$$\arg\left(\frac{\lambda(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu+\alpha}\right) \leq (1+\mu)\arg(\lambda).$$

Therefore, if $\arg(\lambda) < \frac{\pi}{2}$, then proceeding as above we get that

$$\arg\left(\frac{\lambda(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu+\alpha}\right) < (1+\mu)\frac{\pi}{2} < \phi < \frac{\pi}{2}+\theta.$$

Using (4.6) again, we can then conclude that $h(\lambda) \in \omega + \Sigma_\theta$ and the proof of the claim is finished.

We have shown that the representation of $S^\mu_{\alpha,\beta}(t)$ (and therefore the representation of $G^\mu_{\alpha,\beta}(t)$ given in (4.4)) is meaningful.

Next, we split the path γ into two parts γ_1 and γ_2 whose supports Γ_1 and Γ_2 are the sets formed by the complex numbers λ such that $\lambda^{\mu+1}$ lies on the intersection of Γ and the boundaries of $\frac{1}{g(t)} + \Sigma_\theta$ and S_ϕ respectively, that is,

$$\Gamma_1 = \Gamma \cap \overline{\left\{\frac{1}{g(t)}+\Sigma_\theta\right\}} \quad \text{and} \quad \Gamma_2 = \Gamma \cap \overline{\{S_\phi\}}.$$

Thus, $\Gamma = \Gamma_1 \cup \Gamma_2$ and $G^\mu_{\alpha,\beta}(t) = I_1(t) + I_2(t)$, for $t > 0$, where

$$I_j(t) := \frac{1}{2\pi i}\int_{\gamma_j} e^{\lambda t}\frac{\lambda^\mu}{\lambda^\mu+\alpha}\left(\frac{(\lambda-\beta)\lambda^\mu}{\lambda^\mu+\alpha}I - A\right)^{-1}d\lambda, \quad j = 1, 2.$$

Next, we estimate the norms $\|I_1(t)\|$ and $\|I_2(t)\|$.

Using (4.3) and (4.4) we get that for every $t > 0$,

$$\|I_1(t)\| \leq \frac{M}{2\pi}\int_{\gamma_1}|e^{\lambda t}|\left|\frac{\lambda^\mu}{\lambda^{\mu+1}-(\beta+\omega)\lambda^\mu-\alpha\omega}\right||d\lambda|. \tag{4.7}$$

Let $\lambda_{\min}$ be the complex $\lambda \in \mathbb{C}$ such that $\text{Im}(\lambda) > 0$, and $|\lambda^{\mu+1}-(\beta+\omega)\lambda^\mu-\alpha\omega| = \text{dist}(L,\alpha\omega)$, where L in the line passing through the point $(\frac{1}{g(t)},0)$ and the intersection of Γ_1 and Γ_2. Then, for $\lambda \in \Gamma_1$ we have that

$$\frac{1}{|\lambda^{\mu+1}-(\beta+\omega)\lambda^\mu-\alpha\omega|} \leq \frac{g(t)}{\cos(\theta)(1+\alpha|\omega|g(t))}, \quad t > 0. \tag{4.8}$$

It follows from (4.7) and (4.8) that for every $t > 0$,

$$\|I_1(t)\| \leq \frac{M}{2\pi}\frac{g(t)}{\cos(\theta)(1+\alpha|\omega|g(t))}\int_{\gamma_1}|e^{\lambda t}||\lambda|^\mu|d\lambda|. \tag{4.9}$$

Recall that

$$|\arg(\lambda^{\mu+1})| < \phi < \frac{\pi}{2}+\theta \leq (1+\widetilde{\mu})\frac{\pi}{2}.$$

Therefore the contour can be chosen such that

$$(\mu+1)\frac{\pi}{2} < |\arg(\lambda^{\mu+1})| < \phi < \frac{\pi}{2}+\theta \leq (1+\widetilde{\mu})\frac{\pi}{2}.$$

Thus, letting $\varphi = |\arg(\lambda)|$, we have that $\frac{\pi}{2} < \varphi < \frac{1+\widetilde{\mu}}{1+\mu}\frac{\pi}{2}$. It follows from (4.9) that (note that $\cos(\varphi) < 0$) for every $t > 0$,

$$\|I_1(t)\| \leq \frac{M}{2\pi}\frac{g(t)}{\cos(\theta)(1+\alpha|\omega|g(t))}\int_0^\infty e^{s\cos(\varphi)t}s^\mu ds \leq \frac{C}{1+\alpha|\omega|g(t)}. \tag{4.10}$$

Similarly, if $\lambda \in \Gamma_2$, then

$$\frac{1}{|\lambda^{\mu+1} - (\beta+\omega)\lambda^\mu - \alpha\omega|} \leq \frac{1}{|\lambda^{\mu+1} - \alpha\omega|} \leq \frac{g(t)}{|\cos(\theta)|}, \quad t > 0.$$

Hence, there exists a constant $C > 0$ such that

$$\|I_2(t)\| \leq \frac{M}{2\pi} \frac{g(t)}{|\cos(\theta)|} \int_{\gamma_2} |e^{\lambda t}||\lambda|^\mu |d\lambda| \leq Cg(t) \int_0^\infty e^{s\cos(\theta)t} s^\mu ds = C, \quad t > 0. \qquad (4.11)$$

It follows from (4.10) and (4.11) that there exists a constant $C > 0$ such that

$$\|G_{\alpha,\beta}^\mu(t)\| \leq C\left[1 + \frac{1}{1 + \alpha|\omega|g(t)}\right], \quad \forall\, t > 0. \qquad (4.12)$$

The estimate (2.1) follows from (4.2), (4.12) and the strong continuity of the resolvent family on $[0,\infty)$. The proof of part (a) is complete.

(b) Now assume that $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$. We proceed similarly as in part (a) by considering again the same function $g(t) = t^{\mu+1}$, $t > 0$. But here we set

$$\Gamma_1 := \Gamma \cap \overline{\left\{\alpha\omega + \frac{1}{g(t)} + \Sigma_\theta\right\}} \quad \text{and} \quad \Gamma_2 := \Gamma \cap \overline{\{S_\phi\}}.$$

First, we need to show that in this case, $h(\lambda) \in \omega + \Sigma_\theta$ for all $\lambda := re^{i\phi_0}$ in this sector. We proceed similarly as in part (a). If $\mathrm{Re}(\lambda) > 0$, then by Lemma 3.3 $h(\lambda) \in \omega + \Sigma_\theta$ as in the proof of part (a). Now, if $\mathrm{Re}(\lambda) < 0$ and λ belongs to this sector, we take again $z := \lambda + \beta = Re^{i\varphi}$ and we recall that

$$\frac{(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu + \alpha} = \frac{z^\mu}{z^\mu + \alpha} = \frac{R^{2\mu} + \alpha R^\mu \cos(\varphi\mu) + \alpha R^\mu \sin(\varphi\mu)i}{R^{2\mu} + 2\alpha R^\mu \cos(\varphi\mu) + \alpha^2}.$$

Since the assertion is true for all $\mathrm{Re}(\lambda) > 0$, we may assume that $\frac{\pi}{2} < \varphi < \pi$. We claim that

$$\mathrm{Im}\left(\frac{z^\mu}{z^\mu + \alpha}\right) \mathrm{Re}\left(\frac{z^\mu}{z^\mu + \alpha}\right) < 0$$

for all $R \geq 0$. In fact,

$$\mathrm{Im}\left(\frac{z^\mu}{z^\mu + \alpha}\right) \mathrm{Re}\left(\frac{z^\mu}{z^\mu + \alpha}\right) = \frac{\alpha R^{2\mu} \sin(\varphi\mu)[R^\mu + \alpha\cos(\varphi\mu)]}{(R^{2\mu} + 2\alpha R^\mu \cos(\varphi\mu) + \alpha^2)^2}.$$

Since $\alpha < 0$ and $\frac{\pi}{2} < \varphi < \pi$, we have that $R^\mu + \alpha\cos(\varphi\mu) \geq 0$ for all $R \geq 0$. Therefore,

$$\alpha R^{2\mu} \sin(\varphi\mu)[R^\mu + \alpha\cos(\varphi\mu)] < 0$$

and the claim is proved. We can deduce that

$$\frac{\mathrm{Im}\left(\frac{z^\mu}{z^\mu + \alpha}\right)}{\mathrm{Re}\left(\frac{z^\mu}{z^\mu + \alpha}\right)} < 0.$$

Thus,

$$\arg\left(\frac{z^\mu}{z^\mu + \alpha}\right) = \arctan\left(\frac{\mathrm{Im}\left(\frac{z^\mu}{z^\mu + \alpha}\right)}{\mathrm{Re}\left(\frac{z^\mu}{z^\mu + \alpha}\right)}\right) < 0,$$

which implies that

$$\arg\left(\frac{\lambda(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu + \alpha}\right) = \arg(\lambda) + \arg\left(\frac{(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu + \alpha}\right) < \arg(\lambda) < (1+\mu)\arg(\lambda) < \frac{\pi}{2} + \theta.$$

Using (4.6) again, we can conclude that $h(\lambda) \in \omega + \Sigma_\theta$. This implies that the representation (4.4) of $G_{\alpha,\beta}^\mu(t)$ is also valid in this case.

On Γ_1, we have that

$$\left| \frac{1}{\lambda^{\mu+1} - (\omega + \beta)\lambda^\mu - \alpha\omega} \right| \leq \frac{1}{|\lambda^{\mu+1} - \alpha\omega|} \leq \frac{g(t)}{\cos(\theta)}, \quad t > 0, \tag{4.13}$$

and there exists a constant $C > 0$ such that

$$|\lambda|^{\mu+1} \leq C\left(\alpha\omega + \frac{1}{g(t)}\right). \tag{4.14}$$

It follows from (4.14) that on Γ_1, we have that for every $t > 0$,

$$|e^{t\lambda}| \leq e^{t|\lambda|} \leq e^{Ct\left(\alpha\omega + \frac{1}{t^{\mu+1}}\right)^{\frac{1}{\mu+1}}} \leq C e^{(\alpha\omega)^{\frac{1}{\mu+1}} t}. \tag{4.15}$$

Using (4.13), (4.14) and (4.15) we get that for every $t > 0$,

$$\|I_1(t)\| \leq \frac{C}{2\pi} \frac{g(t)}{\cos(\theta)} \left(\alpha\omega + \frac{1}{g(t)}\right) e^{(\alpha\omega)^{\frac{1}{\mu+1}} t} \int_{\gamma_1} \frac{1}{|\lambda|} |d\lambda|. \tag{4.16}$$

Since

$$\text{lenght}(\gamma_1) \leq C\left(\alpha\omega + \frac{1}{g(t)}\right)^{\frac{1}{\mu+1}}, \quad t > 0,$$

and on Γ_1 (by using the law of sines),

$$|\lambda| \geq \left[\cos(\theta)\left(\alpha\omega + \frac{1}{g(t)}\right)\right]^{\frac{1}{\mu+1}}, \quad t > 0,$$

it follows from (4.16) that for every $t > 0$,

$$\|I_1(t)\| \leq C\left[1 + \alpha\omega g(t)\right] e^{(\alpha\omega)^{\frac{1}{\mu+1}} t}. \tag{4.17}$$

Next, we consider the integral $I_2(t)$. Let z_t and $\bar{z}_t$ be the intersection points of the boundary of $\alpha\omega + \frac{1}{g(t)} + \Sigma_\theta$ and S_ϕ, for which we have

$$|\lambda^{\mu+1} - (\omega + \beta)\lambda^\mu - \alpha\omega| \geq |\lambda^{\mu+1} - \alpha\omega| \geq |\cos(\phi)||z_t|, \quad \lambda \in \Gamma_2,$$

and

$$|z_t| \geq C\left(\alpha\omega + \frac{1}{g(t)}\right), \quad t > 0.$$

The same bounds are also valid for the conjugate $\bar{z}_t$ of z_t. Letting $\gamma_2(s) = se^{i\varphi}$, $s > 0$ (recall that $\cos(\varphi) < 0$), we get that for every $t > 0$,

$$\begin{aligned}
\|I_2(t)\| \leq &\frac{1}{2\pi} \int_{\gamma_2} |e^{\lambda t}| \left| \frac{\lambda^\mu}{\lambda^{\mu+1} - (\omega + \beta)\lambda^\mu - \alpha\omega} \right| |d\lambda| \\
\leq &\frac{1}{2\pi \sin(\phi)|z_t|} \int_{\gamma_2} |e^{\lambda t}||\lambda^\mu| \, |d\lambda| \\
\leq &\frac{Ct^{\mu+1}}{1 + \alpha\omega g(t)} \int_0^\infty e^{t\cos(\varphi)s} s^\mu \, ds \\
\leq &\frac{C}{1 + \alpha\omega g(t)}. \tag{4.18}
\end{aligned}$$

It follows from (4.17) and (4.18) that for every $t > 0$,

$$\|G^{\mu}_{\alpha,\beta}(t)\| \leq C \left[1 + \alpha\omega g(t)\right] e^{(\alpha\omega)^{\frac{1}{\mu+1}} t}. \tag{4.19}$$

Now, the estimate (2.2) follows from (4.2), (4.19) and the strong continuity of the resolvent family on $[0, \infty)$. The proof is finished. $\qquad\square$

Proof of Theorem 2.4. Let $\alpha > 0$, $\beta \geq 0$, $0 < \mu < 1$ and $\omega < 0$. Assume that $\beta + \omega > 0$. By using the same notation as in the proof of Theorem 2.3 (part (a)), we have that

$$I_j(t) = \frac{1}{2\pi i} \int_{\gamma_j} e^{\lambda t} \frac{\lambda^{\mu}}{\lambda^{\mu} + \alpha} \left(\frac{(\lambda - \beta)\lambda^{\mu}}{\lambda^{\mu} + \alpha} I - A\right)^{-1} d\lambda, \quad j = 1, 2, \tag{4.20}$$

where $\gamma := \gamma_1 + \gamma_2$ is the contour defined in the proof of Theorem 2.3 (part (a)). We recall that

$$\Gamma_1 = \Gamma \cap \overline{\left\{\frac{1}{g(t)} + S_\theta\right\}} \quad \text{and} \quad \Gamma_2 = \Gamma \cap \overline{\{S_\phi\}},$$

Now, we fix $t > \beta$. Since A is ω-sectorial and $\alpha\omega < 0$, we have that the integral $I_1(t)$ is bounded as follows:

$$\|I_1(t)\| \leq \frac{M}{2\pi} \int_{\gamma_1} |e^{\lambda t}| \left|\frac{\lambda^{\mu}}{\lambda^{\mu+1} - (\beta + \omega)\lambda^{\mu} - \alpha\omega}\right| |d\lambda|$$

$$\leq \frac{M}{2\pi} \int_{\gamma_1} |e^{\lambda t}| \left|\frac{\lambda^{\mu}}{\lambda^{\mu+1} - (\beta + \omega)\lambda^{\mu}}\right| |d\lambda|$$

$$= \frac{M}{2\pi} \int_{\gamma_1} |e^{\lambda t}| \frac{1}{|\lambda - (\beta + \omega)|} |d\lambda|.$$

Since $\gamma_1(s) = se^{i\varphi}$, it follows that for $t > \beta$ (recall that $\cos(\varphi) < 0$)

$$\|I_1(t)\| \leq \frac{M}{\pi} \int_0^{\infty} \frac{e^{s\cos(\varphi)t}}{\sqrt{(s - \cos(\varphi)(\beta + \omega))^2 + (\beta + \omega)^2 \sin^2(\varphi)}} ds$$

$$= \frac{M e^{\cos^2(\varphi)(\beta+\omega)t}}{\pi} \int_{-\cos(\varphi)(\beta+\omega)}^{\infty} \frac{e^{\tau \cos(\varphi)t}}{\sqrt{\tau^2 + (\beta + \omega)^2 \sin^2(\varphi)}} d\tau$$

$$\leq \frac{M e^{(\beta+\omega)t}}{\pi(\beta + \omega)\sin(\varphi)} \int_0^{\infty} e^{\tau \cos(\varphi)\beta} d\tau$$

$$\leq C e^{(\beta+\omega)t}. \tag{4.21}$$

Proceeding similarly, we get that there exists a constant $C_2 > 0$ such that

$$\|I_2(t)\| \leq C_2 e^{(\beta+\omega)t}, \quad \forall\, t > \beta. \tag{4.22}$$

If $t \in [0, \beta]$, then the strong continuity of $S^{\mu}_{\alpha,\beta}(t)$ implies that $G^{\mu}_{\alpha,\beta}(t)$ is strongly continuous for all $t \in [0, \beta]$. Using the uniform boundedness principle and the compactness of $[0, \beta]$, we can conclude that $G^{\mu}_{\alpha,\beta}(t)$ is uniformly bounded in $[0, \beta]$, that is,

$$\|G^{\mu}_{\alpha,\beta}(t)\| \leq C_3, \quad t \in [0, \beta]. \tag{4.23}$$

Since $\beta + \omega > 0$ by assumption, combining (4.21)-(4.22) and (4.23), we can deduce that there exists a constant $C > 0$ such that

$$\|G^{\mu}_{\alpha,\beta}(t)\| \leq C e^{(\beta+\omega)t}, \quad \forall\, t \geq 0.$$

We have shown the estimate (2.3) and the proof is finished. $\qquad\square$

5. Explicit representation of resolvent families

In this section, we consider particular examples of operators A and give a more explicit (than the one in (1.5)) representation of the resolvent family associated with the system (1.1) in terms of some special functions and we also investigate their precise exponential bound.

5.1. **The case** $A = \rho I$. Here, we assume that the operator A is given by $A = \rho I$ for some $\rho \in \mathbb{R}$. Hence, the system (1.1) becomes

$$\begin{cases} u'(t) = \rho u(t) + \dfrac{\rho \alpha}{\Gamma(\mu)} \displaystyle\int_0^t e^{-\beta(t-s)} (t-s)^{\mu-1} u(s) ds, & t > 0, \\ u(0) = u_0. \end{cases} \tag{5.1}$$

We have the following explicit representation of solutions.

Proposition 5.1. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu < 1$, $\beta \geq 0$ and $A = \rho I$ for some $\rho \in \mathbb{R}$. Assume that $\alpha > 0$, or $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$. Then the strongly continuous exponentially bounded resolvent family $S_{\alpha,\beta}^\mu$ associated with the system (5.1) is given by*

$$S_{\alpha,\beta}^\mu(t) = e^{-\beta t} \sum_{k=0}^\infty (\rho + \beta)^k t^k E_{\mu+1,k+1}^{(k+1)}(\alpha \rho t^{\mu+1}), \quad t \geq 0, \tag{5.2}$$

provided that the series converges, and where

$$E_{\mu+1,k+1}^{(k+1)}(z) := \sum_{n=0}^\infty \frac{(k+n)! z^n}{n! k! \Gamma(n\mu + k + n + 1)}, \quad z \in \mathbb{C}, \tag{5.3}$$

denotes the generalized Mittag-Leffler function.

Proof. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu < 1$, $\beta \geq 0$ and $A = \rho I$ for some $\rho \in \mathbb{R}$. Assume that $\alpha > 0$, or $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$. It follows from Corollary 2.2 that there exists a strongly continuous resolvent family $S_{\alpha,\beta}^\mu$ of type $(M, \widetilde{\omega})$ (for some $\widetilde{\omega} > 0$) such that the unique solution of (5.1) is given by (1.5). In addition, we have that the Laplace transform of $S_{\alpha,\beta}^\mu(t)$ is given by

$$\widehat{S}_{\alpha,\beta}^\mu(\lambda) := \frac{(\lambda+\beta)^\mu}{(\lambda+\beta)^{\mu+1} - (\rho+\beta)(\lambda+\beta)^\mu - \alpha\rho}, \quad \text{Re}(\lambda) > \widetilde{\omega}. \tag{5.4}$$

Using the properties of the Laplace transform, we have that

$$S_{\alpha,\beta}^\mu(t) = e^{-\beta t} G_{\alpha,\beta}^\mu(t), \tag{5.5}$$

for some function $G_{\alpha,\beta}^\mu(t)$ whose Laplace transform is given for $\text{Re}(\lambda) > \widetilde{\omega} + \beta$ by

$$\widehat{G}_{\alpha,\beta}^\mu(\lambda) = \frac{\lambda^\mu}{\lambda^{\mu+1} - (\rho+\beta)\lambda^\mu - \alpha\rho}. \tag{5.6}$$

Since

$$\left| \frac{(\rho+\beta)\lambda^\mu}{\lambda^{\mu+1} - \alpha\rho} \right| < 1,$$

for $|\lambda|$ large enough, we have that

$$\widehat{G}^{\mu}_{\alpha,\beta}(\lambda) = \frac{\lambda^{\mu}}{\lambda^{\mu+1} - (\rho+\beta)\lambda^{\mu} - \alpha\rho} = \frac{\lambda^{\mu}}{[\lambda^{\mu+1} - \alpha\rho][1 - \frac{(\rho+\beta)\lambda^{\mu}}{\lambda^{\mu+1} - \alpha\rho}]}$$

$$= \frac{\lambda^{\mu}}{\lambda^{\mu+1} - \alpha\rho} \sum_{k=0}^{\infty} \frac{(\rho+\beta)^k \lambda^{k\mu}}{[\lambda^{\mu+1} - \alpha\rho]^k} = \sum_{k=0}^{\infty} \frac{(\rho+\beta)^k \lambda^{(k+1)\mu}}{[\lambda^{\mu+1} - \alpha\rho]^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{(\rho+\beta)^k \lambda^{2(k+1)\mu}}{\lambda^{(k+1)\mu}[\lambda^{\mu+1} - \alpha\rho]^{k+1}}.$$

Taking the inverse Laplace transform, we get that (see e.g. [11, Formula 17.6])

$$\mathcal{L}^{-1}\left(\frac{\lambda^{2(k+1)\mu}}{\lambda^{(k+1)\mu}[\lambda^{\mu+1} - \alpha\rho]^{k+1}}\right)(t) = t^k E^{(k+1)}_{\mu+1,k+1}(\alpha\rho t^{\mu+1}),$$

where $E^{(k+1)}_{\mu+1,k+1}$ is the generalized Mittag-Leffler function given in (5.3). We have shown that

$$G^{\mu}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} (\rho+\beta)^k t^k E^{(k+1)}_{\mu+1,k+1}(\alpha\rho t^{\mu+1}). \tag{5.7}$$

Now (5.2) follows from (5.5) and (5.7). The proof is finished. $\qquad\square$

We have the following result as a direct consequence of Theorem 2.3.

Corollary 5.2. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu < 1$, $\beta \geq 0$, $A = \rho I$ for some $\rho < 0$ and assume in addition that $\rho + \beta \leq 0$. Then the following assertions hold.*

(a) *If $\alpha > 0$, then there exists a constant $M > 0$ such that*

$$\|S^{\mu}_{\alpha,\beta}(t)\| \leq Me^{-\beta t}, \quad \forall\, t \geq 0. \tag{5.8}$$

(b) *If $\alpha < 0$ and $\alpha + \beta^{\mu} \geq |\alpha|$, then there exists a constant $M > 0$ such that*

$$\|S^{\mu}_{\alpha,\beta}(t)\| \leq M\left(1 + \alpha\rho t^{\mu+1}\right) e^{-(\beta - (\alpha\rho)^{\frac{1}{\mu+1}})t}, \quad \forall\, t \geq 0. \tag{5.9}$$

Proof. This is a particular case of Theorem 2.3, since A is ω-sectorial of angle θ for every $0 < \theta < \frac{\pi}{2}$, with $\omega = \rho < 0$ and $\omega + \beta = \rho + \beta \leq 0$ by assumption. The proof is finished. $\qquad\square$

In the following remark, we show that if $\mu = 1$, then one recovers some of the results obtained in the references [2, 15].

Remark 5.3. Let $\mu = 1$ and $A = \rho I$ for some $\rho < 0$. It follows from (5.6) that

$$\widehat{G}^{1}_{\alpha,\beta}(\lambda) = \frac{\lambda}{\lambda^2 - (\rho+\beta)\lambda - \alpha\rho}.$$

Let

$$D := (\rho+\beta)^2 + 4\alpha\rho.$$

We have the following three situations.

(i) If $D > 0$, then let $\lambda_1 := \frac{\rho+\beta+\sqrt{D}}{2}$ and $\lambda_2 := \frac{\rho+\beta-\sqrt{D}}{2}$. Using partial fractions, we get that

$$\widehat{G}^{1}_{\alpha,\beta}(\lambda) = \frac{1}{\sqrt{D}}\left(\frac{\lambda_1}{\lambda - \lambda_1} - \frac{\lambda_2}{\lambda - \lambda_2}\right).$$

This implies that

$$G^{1}_{\alpha,\beta}(t) = \frac{\rho+\beta+\sqrt{D}}{2\sqrt{D}} e^{\frac{\rho+\beta+\sqrt{D}}{2}t} - \frac{\rho+\beta-\sqrt{D}}{2\sqrt{D}} e^{\frac{\rho+\beta-\sqrt{D}}{2}t}, \quad \forall\, t \geq 0,$$

so that

$$S^1_{\alpha,\beta}(t) = \frac{\rho + \beta + \sqrt{D}}{2\sqrt{D}}e^{\frac{\rho-\beta+\sqrt{D}}{2}t} - \frac{\rho + \beta - \sqrt{D}}{2\sqrt{D}}e^{\frac{\rho-\beta-\sqrt{D}}{2}t}, \ \forall \, t \geq 0.$$

In that case, one has uniform exponential stability if and only if

$$\rho - \beta + \sqrt{(\rho+\beta)^2 + 4\alpha\rho} < 0.$$

(ii) If $D = 0$, then

$$\widehat{G}^1_{\alpha,\beta}(\lambda) = \frac{\lambda}{\left(\lambda - \frac{\rho+\beta}{2}\right)^2} = \frac{1}{\lambda - \frac{\rho+\beta}{2}} + \frac{\rho+\beta}{2}\frac{1}{\left(\lambda - \frac{\rho+\beta}{2}\right)^2}.$$

This implies that

$$G^1_{\alpha,\beta}(t) = \left(1 + \frac{\rho+\beta}{2}t\right)e^{\frac{\rho+\beta}{2}t}, \ \forall \, t \geq 0,$$

so that

$$S^1_{\alpha,\beta}(t) = \left(1 + \frac{\rho+\beta}{2}t\right)e^{\frac{\rho-\beta}{2}t}, \ \forall \, t \geq 0.$$

In that case, one has uniform exponential stability if and only if

$$\rho - \beta < 0.$$

(iii) Now if $D < 0$, then let $\lambda_0 := \frac{\rho+\beta+i\sqrt{-D}}{2}$ and $\overline{\lambda_0} = \frac{\rho+\beta-i\sqrt{-D}}{2}$. Proceeding as above we get that for every $t \geq 0$,

$$S^1_{\alpha,\beta}(t) = \left(\cos\left(\sqrt{-D}t\right) - i\frac{\rho+\beta}{\sqrt{-D}}\sin\left(\sqrt{-D}t\right)\right)e^{\frac{\rho-\beta}{2}t},$$

so that, one has uniform exponential stability if and only if

$$\rho - \beta < 0.$$

5.2. **The case of a self-adjoint operator.** We assume that $-A$ is a non-negative and self-adjoint operator with compact resolvent on the Hilbert space $L^2(\Omega)$ where $\Omega \subset \mathbb{R}^N$ is a bounded open set. Since $-A$ is non-negative and self-adjoint with compact resolvent, we can deduce that it has a discrete spectrum which is formed of eigenvalues. Its eigenvalues satisfy $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ and $\lim_{n\to\infty} \lambda_n = \infty$. We denote the normalized eigenfunction associated with λ_n by ϕ_n. Then $\{\phi_n : n \in \mathbb{N}\}$ is an orthonormal basis for $L^2(\Omega)$ and the set is also total in $D(A)$. Observe also that for every $v \in D(A)$ we can write

$$-Av = \sum_{k=1}^{\infty} \lambda_n \langle v, \phi_n \rangle_{L^2(\Omega)} \phi_n.$$

We have the following result as a direct consequence of Proposition 5.1.

Corollary 5.4. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu < 1$, $\beta > 0$ and let A be as above. Assume that $\alpha > 0$, or $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$. Then the strongly continuous exponentially bounded resolvent family $S^\mu_{\alpha,\beta}$ associated with the system (1.1) is given by*

$$S^\mu_{\alpha,\beta}(t) = e^{-\beta t}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(\beta - \lambda_n)^k t^k E^{(k+1)}_{\mu+1,k+1}(-\alpha\lambda_n t^{\mu+1}), \ \ t \geq 0, \tag{5.10}$$

provided that the series converges.

Proof. Let A be as above. Multiplying both sides of (1.1) by $\phi_n(x)$ and integrating over the set Ω we get that for every $n \in \mathbb{N}$, the function $u_n(t) := \langle u(t), \phi_n \rangle_{L^2(\Omega)}$ is a solution of the system

$$\begin{cases} u_n'(t) = -\lambda_n u_n(t) + \dfrac{-\lambda_n \alpha}{\Gamma(\mu)} \displaystyle\int_0^t e^{-\beta(t-s)}(t-s)^{\mu-1} u_n(s)ds, \ t > 0 \\ u_n(0) = u_{0,n}, \end{cases} \tag{5.11}$$

where $u_{0,n} = \langle u_0, \phi_n \rangle_{L^2(\Omega)}$. It follows from Proposition 5.1 that

$$u_n(t) = S^\mu_{\alpha,\beta,n}(t)u_{0,n}, \ \forall \, t \geq 0, \tag{5.12}$$

where for every $n \in \mathbb{N}$,

$$S^\mu_{\alpha,\beta,n}(t) = e^{-\beta t} \sum_{k=0}^\infty (\beta - \lambda_n)^k t^k E^{(k+1)}_{\mu+1,k+1}(-\alpha\lambda_n t^{\mu+1}), \ \forall \, t \geq 0. \tag{5.13}$$

Since

$$u(t,x) = \sum_{n=1}^\infty u_n(t)\phi_n(x), \ \forall \, t \geq 0, \ x \in \Omega,$$

we have that

$$u(t,x) = \sum_{n=1}^\infty S^\mu_{\alpha,\beta,n}(t)u_{0,n}\phi_n(x), \ \ \forall \, t \geq 0, \ x \in \Omega, \tag{5.14}$$

so that $S^\mu_{\alpha,\beta}$ is given by the expression in (5.10). The proof is finished. $\qquad\square$

We conclude the paper with the following result.

Corollary 5.5. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $0 < \mu < 1$, $\beta \geq 0$ and let A be as above. Assume that the first eingenvalue $\lambda_1 > 0$ and that $\beta - \lambda_1 \leq 0$. Then the following assertions hold.*

(a) *If $\alpha > 0$, then there exists a constant $M > 0$ such that*

$$\|S^\mu_{\alpha,\beta}(t)\| \leq Me^{-\beta t}, \quad \forall \, t \geq 0. \tag{5.15}$$

(b) *If $\alpha < 0$ and $\alpha + \beta^\mu \geq |\alpha|$, then there exists a constant $M > 0$ such that*

$$\|S^\mu_{\alpha,\beta}(t)\| \leq M\left(1 - \alpha\lambda_1 t^{\mu+1}\right) e^{-(\beta - (-\alpha\lambda_1)^{\frac{1}{\mu+1}})t}, \quad \forall \, t \geq 0. \tag{5.16}$$

Proof. This is also a particular case of Theorem 2.3, given that by assumption, the operator A is ω-sectorial of angle θ, for every $0 < \theta < \frac{\pi}{2}$, with $\omega = -\lambda_1 < 0$, and that $\beta + \omega = \beta - \lambda_1 \leq 0$. The proof is finished. $\qquad\square$

Acknowledgments

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RODRIGO PONCE, Universidad de Talca, Instituto de Matemáticas, CASILLA 747, TALCA-CHILE.
E-mail address: rponce@inst-mat.utalca.cl

MAHAMADI WARMA, DEPARTMENT OF MATHEMATICAL SCIENCES AND THE CENTER FOR MATHEMATICS AND ARTIFICIAL INTELLIGENCE (CMAI), GEORGE MASON UNIVERSITY, VA 22030, USA.
E-mail address: mwarma@gmu.edu